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Velocity, attenuation, and quality factor in anisotropic viscoelastic media: A perturbation approach

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ABSTRACT

Velocity, attenuation, and the quality (Q-) factor of waves propagating in homogeneous media of arbitrary anisotropy and attenuation strength are calculated in high-frequency asymptotics using a stationary slowness vector, the vector evaluated at the stationary point of the slowness surface. This vector is generally complex-valued and inhomogeneous, meaning that the real and imaginary parts of the vector have different directions. The slowness vector can be determined by solving three coupled polynomial equations of the sixth order or by a nonlinear inversion. The procedure is simplified if perturbation theory is applied. The elastic medium is viewed as a background medium, and the attenuation effects are incorporated as perturbations. In the first-order approximation, the phase and ray velocities and their directions remain unchanged, being the same as those in the background elastic medium. The perturbation of the slowness vector is calculated by solving a system of three linear equations. The phase attenuation and phase Q-factor are linear functions of the perturbation of the slowness vector. Calculating the ray attenuation and ray Q-factor is even simpler than calculating the phase quantities because they are expressed in terms of perturbations of the medium without the need to evaluate the perturbation of the slowness vector. Numerical modeling indicates that the perturbations are highly accurate; the errors are less than 0.3% for a medium with a Q-factor of 20 or higher. The accuracy can be enhanced further by a simple modification of the first-order perturbation formulas.

INTRODUCTION

The high-frequency asymptotic approximation is used frequently in wave-propagation modeling because it is reasonably accurate in many seismic applications and requires almost no computer time compared with exact methods. Asymptotic formulas for velocities, attenuations, and quality (Q-) factors of waves propagating in anisotropic attenuative media have been published (Vavryčuk, 2007a, 2007b). The formulas are valid for waves propagating in homogeneous media of arbitrary anisotropy and attenuation strength. The wave quantities are calculated using a stationary slowness vector that predicts the complex energy velocity vector parallel to a ray. The stationary slowness vector is generally complex-valued and inhomogeneous, and determining it is the most complicated step when calculating the asymptotic wave quantities. The stationary slowness vector can be calculated (1) by a nonlinear inversion for four real-valued angles defining the directions of the real and imaginary parts of the slowness vector or (2) by solving a system of three coupled polynomial equations of the sixth order in three unknown components of the complex-valued slowness vector (Vavryčuk, 2006).

The problem is simplified if perturbation theory is applied. The elastic medium is viewed as the background medium, and the attenuation effects are calculated as perturbations. The procedure is similar to that for calculating wave quantities in weakly anisotropic elastic media, in which weak anisotropy is introduced as a perturbation of the isotropic background (Thomsen, 1986; Jech and Pšenčík, 1989; Tsvankin and Thomsen, 1994; Vavryčuk, 1997, 2003; Farra, 2001, 2004; Song et al., 2001; Pšenčík and Vavryčuk, 2002). The only difference is that the isotropic background is degenerate and the perturbations caused by weak anisotropy are real-valued. For anisotropic media with attenuation, the background medium is nondegenerate and perturbations are complex-valued. This approach has been adopted by several authors and applied to various aspects of studying plane-wave propagation in seismic exploration (Carcione, 2000; Chichinina et al., 2006; Zhu and Tsvankin, 2006, 2007; Červený and Pšenčík, 2007).

In this paper, I apply the first-order perturbation theory to calculate high-frequency asymptotic quantities in anisotropic viscoelastic media. I derive the perturbation formula for the stationary slowness vector and consequently for other asymptotic quantities such as ray and phase velocities, attenuations, and Q-factors. The accuracy of the perturbations is tested using numerical examples.

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BASIC PERTURBATION FORMULAS

In formulas, the real and imaginary parts of complex-valued quantities are denoted by superscripts R and I, respectively. A complex-conjugate quantity is denoted by an asterisk. The direction of a complex-valued vector $\mathbf{v}$ is calculated as $\mathbf{v}/\sqrt{\mathbf{v}\cdot\mathbf{v}}$, where the dot indicates a scalar product (the normalization condition $\mathbf{v}/\sqrt{\mathbf{v}\cdot\mathbf{v}^*}$ is not used). The magnitude of $\mathbf{v}$ is complex-valued and calculated as $\sqrt{\mathbf{v}\cdot\mathbf{v}}$. If any complex-valued vector $\mathbf{v}$ is defined by a real-valued direction, it is called homogeneous; if defined by a complex-valued direction, it is called inhomogeneous. The type of wave (P, S1, S2) is denoted by superscripts $[(1),(2),(3)]$. The background quantity is denoted by superscript zero.

The perturbation of f is denoted by Δf. The symbol $\Delta_a f$ denotes the perturbation of $f(a_{ijk\ell},p_i)$ resulting from the perturbations of medium parameters $a_{ijk\ell}$. The symbol $\Delta_p f$ denotes the perturbation of $f(a_{ijk\ell},p_i)$ from the perturbation of slowness vector $\mathbf{p}$. In formulas, the Einstein summation convention is used for repeated subscripts. Besides the standard four-index notation for viscoelastic parameters $a_{ijk\ell}$ and quality parameters $q_{ijk\ell}$, the two-index Voigt notation A_{MN} and Q_{MN} is used. Voigt notation reduces pairs of indices i,j or k,l into a single index M or N using the following rules: $11\rightarrow 1, 22\rightarrow 2$, $33\rightarrow 3$, $23\rightarrow 4$, $13\rightarrow 5$, and $12\rightarrow 6$.

Definition of the medium

A viscoelastic medium is defined by density-normalized stiffness parameters $a_{ijk\ell} = c_{ijk\ell}/\rho$, which are, in general, frequency dependent and complex-valued. The real and imaginary parts of $a_{ijk\ell}$,

$$a_{ijk\ell}(\omega) = a^R_{ijk\ell} + i a^I_{ijk\ell}, \tag{1}$$

define elastic and viscous properties of the medium. Their ratio, called the matrix of quality parameters,

$$q_{ijk\ell}(\omega) = -\frac{a^R_{ijk\ell}}{a^I_{ijk\ell}} (\text{no summation over repeated indices}), \tag{2}$$

quantifies how attenuative the medium is. The sign in formula 2 depends on the definition of the Fourier transform used to calculate the viscoelastic parameters in the frequency domain. Here, I use the forward Fourier transform with the exponential term $\exp(i\omega t)$. Hence, the quality parameters are defined with the minus sign in formula 2. When using the Fourier transform with the exponential term $\exp(-i\omega t)$, the minus sign must be omitted.

Let us assume that $a_{ijk\ell}$ satisfy symmetry relations

$$a_{ijk\ell} = a_{jik\ell} = a_{ij\ell k} = a_{k\ell ij} \tag{3}$$

and that viscous parameters $a^I_{ijk\ell}$ are small with respect to elastic parameters $a^R_{ijk\ell}$. The viscoelastic medium then can be viewed as a medium obtained by a small perturbation of an elastic background,

$$a_{ijk\ell} = a^0_{ijk\ell} + \Delta a_{ijk\ell}, \tag{4}$$

where $a^0_{ijk\ell}$ defines the background medium and $\Delta a_{ijk\ell}$ its perturbation,

$$a^0_{ijk\ell} = a^R_{ijk\ell}, \quad \Delta a_{ijk\ell} = i a^I_{ijk\ell}. \tag{5}$$

Eigenvalues and eigenvectors of the Christoffel tensor

The Christoffel tensor in the viscoelastic medium is defined alternatively in terms of slowness direction $\mathbf{n}$,

$$\Gamma_{jk}(\mathbf{n}) = a_{ijk\ell} n_i n_\ell, \tag{6}$$

or slowness vector $\mathbf{p}$,

$$\Gamma_{jk}(\mathbf{p}) = a_{ijk\ell} p_i p_\ell, \tag{7}$$

where $a_{ijk\ell}$ are the complex-valued viscoelastic parameters. Direction $\mathbf{n}$ and the slowness magnitude $p = \sqrt{p_j p_j}$ are generally complex valued. The Christoffel tensor Γ_{jk} has three eigenvalues and three eigenvectors, which define properties of the P, S1, and S2 waves. Eigenvalues $G(\mathbf{n})$ and $G(\mathbf{p})$ read

$$G(\mathbf{n}) = a_{ijk\ell} n_i n_\ell g_j g_k = c^2, \tag{8}$$

$$G(\mathbf{p}) = a_{ijk\ell} p_i p_\ell g_j g_k = 1, \tag{9}$$

where $c = 1/p$ is the complex phase velocity. The eigenvectors of Γ_{jk} define the polarization vectors $\mathbf{g}$.

If not specified explicitly, the Christoffel tensor Γ_{jk} and eigenvalues G are assumed to be functions of the slowness vector $\mathbf{p}$, $\Gamma_{jk} = \Gamma_{jk}(\mathbf{p})$, and $G = G(\mathbf{p})$. Using perturbation theory, the Christoffel tensor Γ_{jk} is decomposed as

$$\Gamma_{jk} = \Gamma^0_{jk} + \Delta\Gamma_{jk}, \tag{10}$$

where Γ^0_{jk} is the Christoffel tensor in the background medium

$$\Gamma^0_{jk} = a^0_{ijk\ell} p^0_i p^0_\ell \tag{11}$$

and $\Delta\Gamma_{jk}$ is its perturbation controlled by perturbations of elastic parameters $\Delta a_{ijk\ell}$ and by perturbation of the slowness vector $\Delta\mathbf{p}$:

$$\Delta\Gamma_{jk} = \Delta a_{ijk\ell} p^0_i p^0_\ell + a^0_{ijk\ell}(p^0_i \Delta p_\ell + p^0_\ell \Delta p_i). \tag{12}$$

If the Christoffel tensor Γ^0_{jk} has three different eigenvalues $G^{0(m)}$, $m = 1,2,3$, it is called nondegenerate, and standard perturbation formulas for calculating its eigenvalues and eigenvectors can be applied (Korn and Korn, 2000, their section 15.4–11). The perturbations of the P-wave eigenvalue $G^{(1)}$ and the P-wave eigenvector $\mathbf{g}^{(1)}$ of Γ_{jk} are expressed as follows:

$$\Delta G^{(1)} = \Delta\Gamma_{jk} g^{0(1)}_j g^{0(1)}_k, \tag{13}$$

$$\Delta g^{(1)}_i = \frac{\Delta G^{(12)}}{G^{0(1)} - G^{0(2)}} g^{0(2)}_i + \frac{\Delta G^{(13)}}{G^{0(1)} - G^{0(3)}} g^{0(3)}_i, \tag{14}$$

where

$$\Delta G^{(rs)} = \Delta\Gamma_{jk} g^{0(r)}_j g^{0(s)}_k. \tag{15}$$

Formulas for the perturbations $\Delta G^{(2)}$, $\Delta G^{(3)}$, $\Delta \mathbf{g}^{(2)}$, and $\Delta \mathbf{g}^{(3)}$ are analogous.

Energy velocity

The perturbation of energy velocity $\mathbf{v}$ (called the group velocity in elastic media),

$$v_i = \frac{1}{2}\frac{\partial G}{\partial p_i} = a_{ijk\ell} p_\ell g_j g_k, \tag{16}$$

is expressed as

$$\Delta v_i = \Delta a_{ijk\ell} p_\ell^0 g_j^0 g_k^0 + a_{ijk\ell}^0 p_\ell^0 (g_j^0 \Delta g_k + g_k^0 \Delta g_j) + a_{ijk\ell}^0 \Delta p_\ell g_j^0 g_k^0. \tag{17}$$

Taking into account equation 14, we obtain for the P-wave

$$\Delta v_i^{(1)} = \Delta_a v_i^{(1)} + \Delta_p v_i^{(1)}, \tag{18}$$

$$\Delta_a v_i^{(1)} = \Delta a_{ijk\ell} p_\ell^{0(1)} g_j^{0(1)} g_k^{0(1)} + \frac{v_i^{0(12)} \Delta_a G^{(12)}}{G^{0(1)} - G^{0(2)}} + \frac{v_i^{0(13)} \Delta_a G^{(13)}}{G^{0(1)} - G^{0(3)}}, \tag{19}$$

$$\Delta_p v_i^{(1)} = a_{ijk\ell}^0 \Delta p_\ell^{(1)} g_j^{0(1)} g_k^{0(1)} + \frac{v_i^{0(12)} \Delta_p G^{(12)}}{G^{0(1)} - G^{0(2)}} + \frac{v_i^{0(13)} \Delta_p G^{(13)}}{G^{0(1)} - G^{0(3)}}, \tag{20}$$

where

$$v_i^{0(rs)} = a_{ijk\ell}^0 p_\ell^{0(1)} (g_j^{0(r)} g_k^{0(s)} + g_k^{0(r)} g_j^{0(s)}), \tag{21}$$

$$\Delta_a G^{(rs)} = \Delta a_{ijk\ell} p_i^{0(1)} p_\ell^{0(1)} g_j^{0(r)} g_k^{0(s)}, \tag{22}$$

$$\Delta_p G^{(rs)} = a_{ijk\ell}^0 (p_\ell^{0(1)} \Delta p_i^{(1)} + p_i^{0(1)} \Delta p_\ell^{(1)}) g_j^{0(r)} g_k^{0(s)}. \tag{23}$$

Perturbation formulas for the S1- and S2-waves are analogous.

STATIONARY SLOWNESS VECTOR

In this section, the perturbation approach is applied to asymptotic wavefields. The asymptotic quantities describe high-frequency waves observed at large distances from the source and are calculated for the slowness vector, taken at a stationary point on the slowness surface. The stationary point is a point for which the energy velocity vector $\mathbf{v}$ is homogeneous and its direction is parallel to the ray (for details, see Vavryčuk, 2007a). The stationary slowness vector $\mathbf{p}$ is generally complex-valued and inhomogeneous, and it can be calculated by a nonlinear inversion or by solving a system of three coupled sixth-order polynomial equations. Once $\mathbf{p}$ is found, all asymptotic wave quantities can be calculated readily.

If the viscoelastic medium is obtained by a small perturbation of an elastic background medium (see equations 4 and 5), the viscoelastic wave quantities can be calculated as perturbations of the elastic ones by using the general perturbation formulas derived in the previous section. However, the perturbation formulas must be modified further to be valid for asymptotic quantities. The main difference between general perturbations and perturbations for asymptotic quantities lies in the perturbation of slowness vector $\Delta\mathbf{p}$. In general formulas, $\Delta\mathbf{p}$ is not a priori fixed but depends on the studied wave, characterized by its wave normal and wave inhomogeneity (Červený and Pšenčík, 2005) On the other hand, $\Delta\mathbf{p}$ is not arbitrary for asymptotic quantities but takes only one specific value with no freedom in setting the wave normal or wave inhomogeneity. The value of $\Delta\mathbf{p}$ must ensure that $\mathbf{p}$ is taken at the stationary point on the complex-valued slowness surface. Taking the slowness vector at the stationary point implies that $\mathbf{v}$ is homogeneous and parallel to the fixed ray direction.

Perturbation $\Delta\mathbf{v}$

In the first-order perturbation theory, the ray direction $\mathbf{N}$ can be fixed, so energy velocity vector $\mathbf{v}$ is parallel to the ray in the background as well as in the perturbed medium

$$\mathbf{v} = \mathbf{N} v, \quad \mathbf{v}^0 = \mathbf{N} v^0. \tag{24}$$

Consequently, velocity perturbation $\Delta\mathbf{v}$ is parallel to background velocity $\mathbf{v}^0$:

$$\Delta\mathbf{v} = \mathbf{N} \Delta v. \tag{25}$$

Taking into account equation 16 and the fact that the eigenvalue G of Γ_{jk} for the studied wave is equal to one (see equation 9) in the background as well as in the perturbed medium, we obtain

$$v_i p_i = 1, \quad v_i^0 p_i^0 = 1. \tag{26}$$

Hence,

$$\Delta v_i p_i^0 + v_i^0 \Delta p_i = 0. \tag{27}$$

Multiplying equation 17 by p_i^0 and taking into account equation 27, we obtain

$$\Delta v_i p_i^0 = \frac{1}{2} \Delta a_{ijk\ell} p_i^0 p_\ell^0 g_j^0 g_k^0. \tag{28}$$

The left-hand side of equation 28 can be simplified further to read

$$\Delta v_i p_i^0 = \frac{\Delta v}{v^0}, \tag{29}$$

where

$$\Delta v_i = \Delta v N_i \quad \text{and} \quad N_i p_i^0 = \frac{1}{v^0}. \tag{30}$$

Hence,

$$\Delta v = \frac{v^0}{2} \Delta a_{ijk\ell} p_i^0 p_\ell^0 g_j^0 g_k^0 \tag{31}$$

and

$$\Delta v_i = \frac{v_i^0}{2}\Delta a_{mjk\ell}p_m^0 p_\ell^0 g_j^0 g_k^0. \tag{32}$$

Perturbation $\Delta\mathbf{p}$

Equations 18 and 32 yield

$$\Delta_a v_i + \Delta_p v_i = \frac{v_i^0}{2}\Delta a_{mjk\ell}p_m^0 p_\ell^0 g_j^0 g_k^0, \tag{33}$$

where $\Delta_a\mathbf{v}$ and $\Delta_p\mathbf{v}$ are defined in equations 19 and 20. Equation 33 can be expressed as

$$H_{i\ell}^0 \Delta p_\ell = \frac{v_i^0}{2}\Delta a_{mjk\ell}p_m^0 p_\ell^0 g_j^0 g_k^0 - \Delta_a v_i, \tag{34}$$

where $H_{i\ell}^0$ is the wave metric tensor in the background medium (see Vavryčuk, 2003, his equation 15). The wave metric tensor $H_{i\ell}$ is defined as (Červený, 2001, his equation 2.2.65)

$$H_{i\ell}(\mathbf{p}) = \frac{1}{2}\frac{\partial^2 G(\mathbf{p})}{\partial p_i \partial p_\ell} \tag{35}$$

and is connected with the Gaussian curvature of the slowness surface (see Klimeš, 2002). Taking into account equation 15 of Vavryčuk (2003) and equations 19 and 22 in this paper, tensor $H_{i\ell}^0$ and vector $\Delta_a\mathbf{v}$ in equation 34 are expressed for the P-wave as follows:

$$H_{i\ell}^{0(1)} = a_{ijk\ell}^0 g_j^{0(1)} g_k^{0(1)} + \frac{v_i^{0(12)}v_l^{0(12)}}{G^{0(1)} - G^{0(2)}} + \frac{v_i^{0(13)}v_l^{0(13)}}{G^{0(1)} - G^{0(3)}}, \tag{36}$$

$$\Delta_a v_i^{(1)} = \Delta a_{mjk\ell}p_\ell^{0(1)} g_j^{0(1)}\left[\delta_{im}g_k^{0(1)} + \frac{v_i^{0(12)}p_m^{0(1)}g_k^{0(2)}}{G^{0(1)} - G^{0(2)}} + \frac{v_i^{0(13)}p_m^{0(1)}g_k^{0(3)}}{G^{0(1)} - G^{0(3)}}\right], \tag{37}$$

where δ_{ij} is the Kronecker delta. All quantities in the background medium are real valued and $\Delta a_{ijk\ell}$ is purely imaginary, so equations 36 and 37 imply that $H_{i\ell}$ also is real valued and $\Delta_a\mathbf{v}$ is purely imaginary.

Equation 34 represents a system of three linear equations in unknown perturbations Δp_1, Δp_2, and Δp_3; hence, perturbation $\Delta\mathbf{p}$ finally comes out as

$$\Delta p_\ell = [H_{i\ell}^0]^{-1}\left(\frac{v_i^0}{2}\Delta a_{mjk\ell}p_m^0 p_\ell^0 g_j^0 g_k^0 - \Delta_a v_i\right). \tag{38}$$

All background quantities are real valued, and $\Delta a_{ijk\ell}$ and $\Delta_a\mathbf{v}$ are purely imaginary, so $\Delta\mathbf{p}$ is purely imaginary. Obviously, this property is lost when higher-order perturbations are applied. Also the assumption of the fixed N is generally no longer valid in the higher order ray theory.

Note that equation 38 is valid under several limitations. First, viscous parameters must be small with respect to elastic parameters (see equations 4 and 5). Second, the wave must not propagate in directions close to cusp edges on the wave surface. The cusp edges arise when the wavefront is multifolded and correspond to parabolic lines on the slowness surface (Vavryčuk, 2003). The inverse of the wave metric tensor $[H_{i\ell}^0]^{-1}$ is large or diverges near parabolic lines; thus, the perturbation theory is inapplicable. Third, the wave must not propagate near directions associated with singularities (acoustic axes) on the slowness surface (Vavryčuk, 2005). The Christoffel tensor is nearly degenerate in these directions, and the standard perturbation formulas fail.

PHASE VELOCITY, ATTENUATION, AND *Q*-FACTOR

Definitions

Phase quantities describe propagation of a plane tangential to the wavefront. By decomposing $\mathbf{p}$, we obtain (see Vavryčuk, 2007b; his equation 2)

$$\mathbf{p} = [(V^{\text{phase}})^{-1} + iA^{\text{phase}}]\mathbf{s} + iD^{\text{phase}}\mathbf{t}, \tag{39}$$

where V^{phase}, A^{phase}, and D^{phase} are the real-valued phase velocity, phase attenuation, and phase inhomogeneity. Vectors $\mathbf{s}$ and $\mathbf{t}$ are real-valued, mutually perpendicular unit vectors; $\mathbf{s}$ is normal to the wavefront (called the wave normal); and $\mathbf{t}$ lies in the wavefront (called the wave tangent). Hence, the phase velocity and phase attenuation are calculated from $\mathbf{p}$ as

$$V^{\text{phase}} = \frac{1}{|\mathbf{p}^R|}, \tag{40}$$

$$A^{\text{phase}} = \mathbf{p}^I \cdot \mathbf{s}, \tag{41}$$

$$D^{\text{phase}} = \mathbf{p}^I \cdot \mathbf{t}, \tag{42}$$

where

$$\mathbf{s} = \frac{\mathbf{p}^R}{|\mathbf{p}^R|}, \quad \mathbf{t} = \frac{\mathbf{p}^I - (\mathbf{p}^I\cdot\mathbf{s})\mathbf{s}}{|\mathbf{p}^I - (\mathbf{p}^I\cdot\mathbf{s})\mathbf{s}|}, \tag{43}$$

and symbol $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_j a_j}$ denotes the magnitude of real-valued vector $\mathbf{a}$.

The phase Q-factor (Q^{phase}-factor) is defined as (Carcione, 2000, his equation 14; Carcione, 2001, his equation 4.92)

$$Q^{\text{phase}} = -\frac{(c^2)^R}{(c^2)^I}, \tag{44}$$

where c is the complex-valued phase velocity $c = 1/\sqrt{p_i p_i}$.

In elastic media, $\mathbf{p}$ is real-valued. Consequently, the Q^{phase}-factor is infinite, and c becomes real-valued and equals the phase velocity V^{phase}.

Perturbations

Taking into account that

$$\mathbf{p} = \mathbf{p}^0 + \Delta\mathbf{p}, \tag{45}$$

where $\mathbf{p}^0$ is a real-valued vector and perturbation $\Delta\mathbf{p}$ is a purely imaginary vector (see equation 38)

$$\mathbf{p}^0 = \mathbf{p}^{0R}, \quad \Delta\mathbf{p} = i\Delta\mathbf{p}^I, \tag{46}$$

we readily obtain

$$V^{\text{phase}} = (V^{\text{phase}})^0, \tag{47}$$

$$A^{\text{phase}} = \Delta\mathbf{p}^I \cdot \mathbf{s}^0, \tag{48}$$

$$D^{\text{phase}} = |\Delta\mathbf{p}^I - A^{\text{phase}} s^0|. \tag{49}$$

Using the following approximate formula,

$$c^2 = \frac{1}{(p_i^0 + \Delta p_i)(p_i^0 + \Delta p_i)} = \frac{1}{p_i^0 p_i^0 + 2p_i^0 \Delta p_i} = \frac{1}{p_j^0 p_j^0}\left(1 - 2\frac{\Delta p_i p_i^0}{p_j^0 p_j^0}\right), \tag{50}$$

the inverse of the phase quality factor comes out as

$$[Q^{\text{phase}}]^{-1} = 2\frac{\Delta p_i^I p_i^{0R}}{p_j^{0R} p_j^{0R}} = 2c^0 A^{\text{phase}}. \tag{51}$$

Similar to the phase velocity, the wave normal **s** remains unchanged for the background and the perturbed medium.

RAY VELOCITY, ATTENUATION, AND Q-FACTOR

Definitions

Ray quantities describe the propagation of waves along a ray. The ray velocity V^{ray}(see Vavryčuk, 2007b, his equation 21),

$$V^{\text{ray}} = \frac{vv^*}{v^R} = \frac{v^R v^R + v^I v^I}{v^R}, \tag{52}$$

controls the propagation velocity along a ray. The ray attenuation A^{ray} (see Vavryčuk, 2007b, his equation 22),

$$A^{\text{ray}} = -\frac{v^I}{vv^*} = -\frac{v^I}{v^R v^R + v^I v^I}, \tag{53}$$

controls the amplitude decay along a ray. The ray velocity and ray attenuation are real-valued and can be measured in wavefields along a ray. Analogous to the phase Q-factor defined in equation 44, we also can introduce the ray Q-factor (see Vavryčuk, 2007b, his equation 24):

$$Q^{\text{ray}} = -\frac{(v^2)^R}{(v^2)^I}. \tag{54}$$

Introducing ray inhomogeneity D^{ray} makes no sense because **v** is homogeneous; hence, D^{ray} is identically zero.

Perturbations

Substituting the equations

$$\mathbf{v}^R = \mathbf{v}^{0R} \quad \text{and} \quad \Delta\mathbf{v} = i\mathbf{v}^I \tag{55}$$

into equations 52 and 53, taking into account equation 32, and retaining the first-order perturbations, we obtain

$$V^{\text{ray}} = v^{0R} = v^0, \tag{56}$$

$$A^{\text{ray}} = \frac{i\Delta v}{v^{0R} v^{0R}} = -\frac{1}{2v^0}\Delta a^I_{ijk\ell} p_i^0 p_\ell^0 g_j^0 g_k^0. \tag{57}$$

Finally, using equation 31 and the approximate formula

$$v^2 = (v^0 + \Delta v)^2 = (v^0)^2 + 2v^0 \Delta v, \tag{58}$$

the inverse of the ray Q-factor comes out as

$$[Q^{\text{ray}}]^{-1} = -\Delta a^I_{ijk\ell} p_i^0 p_\ell^0 g_j^0 g_k^0 = 2v^0 A^{\text{ray}}. \tag{59}$$

The ray direction **N** is the same for the background and the perturbed media because it was fixed when perturbations were derived. Formulas similar to 57 and 59 are also valid for the ray attenuation and ray Q-factor of weakly inhomogeneous plane waves (see Červený and Pšenčík, 2007). Note that equation 59 also is derived by Gajewski and Pšenčík (1992), who assume a weakly attenuating medium described by Futterman's dispersion relation (Futterman, 1962). Here, I show that equation 59 is valid independent of any type of dispersion.

PHASE AND RAY VELOCITIES WITH IMPROVED ACCURACY

The phase and ray velocities in the perturbed viscoelastic medium have the same value as in the elastic background medium (see equations 47 and 59). This implies that the first-order perturbations do not reflect the velocity shift caused by attenuation of the medium. If such an approximation is insufficient and more accurate formulas are needed, we can combine perturbations with exact calculations. In this way, the perturbations are used only for determining the stationary slowness vector, which is the crucial and most complicated step in the calculations. All other quantities are computed by using exact formulas. Note that this approach is not applicable to inhomogeneous media because the assumption of a fixed ray direction used in perturbations is not exact but approximate for inhomogeneous media.

Specifically, we apply the following procedure. First, the slowness vector **p** is calculated using equations 38 and 45. Second, **p** is normalized to obtain the slowness direction $\mathbf{n} = \mathbf{p}/\sqrt{p_j p_j}$. Third, the complex phase velocity c and real phase velocity V^{phase} are calculated using equations 8 and 40. Finally, the complex energy velocity vector **v** and the ray velocity V^{ray} are calculated using equations 16 and 52.

NUMERICAL EXAMPLES

This section demonstrates the accuracy of the perturbation formulas using numerical examples performed on the P-wave. Eight test

Table 1. Viscoelastic parameters. Two-index Voigt notation is used for density-normalized elastic parameters $a^R_{ijk\ell}$ and quality parameters $q_{ijk\ell}$. Parameter A^R_{66} and Q_{66} are not listed because the P-wave is not sensitive to them.

	Elastic parameters				Attenuation parameters			
Model	A^R_{11} (km²/s²)	A^R_{13} (km²/s²)	A^R_{33} (km²/s²)	A^R_{44} (km²/s²)	Q_{11}	Q_{13}	Q_{33}	Q_{44}
A1	14.4	4.50	9.00	2.25	7.5	4	5	4
A2	14.4	4.50	9.00	2.25	15	8	10	8
A3	14.4	4.50	9.00	2.25	30	16	20	16
A4	14.4	4.50	9.00	2.25	60	32	40	32
B1	10.8	3.53	9.00	2.25	7.5	4	5	4
B2	10.8	3.53	9.00	2.25	15	8	10	8
B3	10.8	3.53	9.00	2.25	30	16	20	16
B4	10.8	3.53	9.00	2.25	60	32	40	32

Table 2. Viscoelastic parameters in Thomsen notation. For a definition of parameters in Thomsen notation, see Thomsen (1986) and Zhu and Tsvankin (2006). Parameters γ and γ_Q are not listed because the P-wave is insensitive to them.

	Elastic parameters				Attenuation parameters			
Model	V_{Po} (km/s)	V_{S0} (km/s)	ε	δ	A_{P0} (10^{-2})	A_{S0} (10^{-2})	ε_Q	δ_Q
A1	3.00	1.50	0.30	0.00	9.90	12.31	−0.333	0.500
A2	3.00	1.50	0.30	0.00	4.99	6.23	−0.333	0.500
A3	3.00	1.50	0.30	0.00	2.50	3.12	−0.333	0.500
A4	3.00	1.50	0.30	0.00	1.25	1.56	−0.333	0.500
B1	3.00	1.50	0.10	−0.10	9.90	12.31	−0.333	0.383
B2	3.00	1.50	0.10	−0.10	4.99	6.23	−0.333	0.383
B3	3.00	1.50	0.10	−0.10	2.50	3.12	−0.333	0.383
B4	3.00	1.50	0.10	−0.10	1.25	1.56	−0.333	0.383

Table 3. P-wave velocity and attenuation anisotropy. The values $\bar{V}^{ray}$, $\bar{A}^{ray}$, and $\bar{Q}^{ray}$ are the mean P-wave ray velocity, attenuation, and Q-factor; a^{ray}_V, a^{ray}_A, and a^{ray}_Q are the P-wave ray velocity anisotropy, attenuation anisotropy, and Q-factor anisotropy. The anisotropy is calculated as $a = 200\,(U^{max} - U^{min})/(U^{max} + U^{min})$, where U^{max} and U^{min} are the maximum and minimum values of the respective quantity.

Model	$\bar{V}^{ray}$ (km/s)	a^{ray}_V (%)	$\bar{A}^{ray}$ (s/km)	a^{ray}_A (%)	$\bar{Q}^{ray}$	a^{ray}_Q (%)
A1	3.32	22.6	28.8×10^{-3}	64.3	5.4	45.4
A2	3.28	23.2	14.7×10^{-3}	65.4	10.8	45.4
A3	3.27	23.3	7.4×10^{-3}	65.6	21.5	45.4
A4	3.27	23.4	3.7×10^{-3}	65.7	43.0	45.4
B1	3.10	9.5	30.2×10^{-3}	53.9	5.4	45.6
B2	3.07	10.3	15.4×10^{-3}	55.1	10.9	45.6
B3	3.06	10.5	7.7×10^{-3}	55.4	21.7	45.6
B4	3.06	10.5	3.9×10^{-3}	55.4	43.5	45.6

viscoelastic models display transverse isotropy with various strength of anisotropy and attenuation. The anisotropy strength ranges from 10% to 20%, close to observations typical for sedimentary rocks (Thomsen, 1986; Vernik and Liu, 1997; Jakobsen and Johansen, 2000). The Q-factors range over a broad interval of values, from moderate (Q = 40–60) to very strong (Q = 5–7) attenuation. The very strong attenuation is rather extreme and typically is not measured (Shankland et al., 1993; Liu et al., 1997; Best et al., 2007) or predicted from theoretical models of sedimentary rocks (Carcione, 2000; Carcione et al., 2003). Here, it demonstrates limits of applicability of perturbations. Although the models are artificial and do not cover all possible variations of anisotropy and attenuation in realistic rocks, they should give sufficient insight into the efficiency of perturbation formulas derived.

The viscoelastic parameters of the models are summarized in Tables 1 and 2. Table 1 lists the parameters in Voigt notation, and Table 2 lists them in Thomsen notation. The eight models combine two models of velocity anisotropy (models A and B) and four levels of attenuation (models A1–A4 and B1–B4). The P-wave anisotropy is about 23% for models A1–A4 and about 10% for models B1–B4 (see Table 3). The strongest attenuation is in models A1 and B1, the mean Q^{ray}-factor being about 5.5. The weakest attenuation is in models A4 and B4, the mean Q^{ray}-factor being about 43.5. The attenuation anisotropy is 65% for models A1–A4 and 55% for models B1–B4. The Q-factor anisotropy is 45.5% for all eight models. The frequency of the signal is assumed to be 30 Hz.

Figure 1 shows polar plots of the P- and SV-wave group velocities for the elastic background of models A and B. The SV-wave group velocity displays a triplication. The triplication causes a nonunique relation between the phase and ray directions. For some ray directions, three wave normals can be observed (see Figure 2). As mentioned, the triplication prevents the perturbation formulas from being applicable; hence, the calculations are performed only for the P-wave.

Figure 3 shows the directional variations of real and imaginary parts of the slowness, the directional variations of the phase and ray velocities, the attenuations, and the Q-factors for model A1. The angles range from 0° to 90°. The slowness, velocities, attenuations, and Q-factors are calculated using two approaches: (1) asymptotic formulas described in Vavryčuk (2007b) and (2) the first-order perturbations derived in this paper. Figure 3 shows that the perturbations yield biased results with errors of about 2%–4%. The best approximation is obtained for the Q-factor, with errors less than 0.5%.

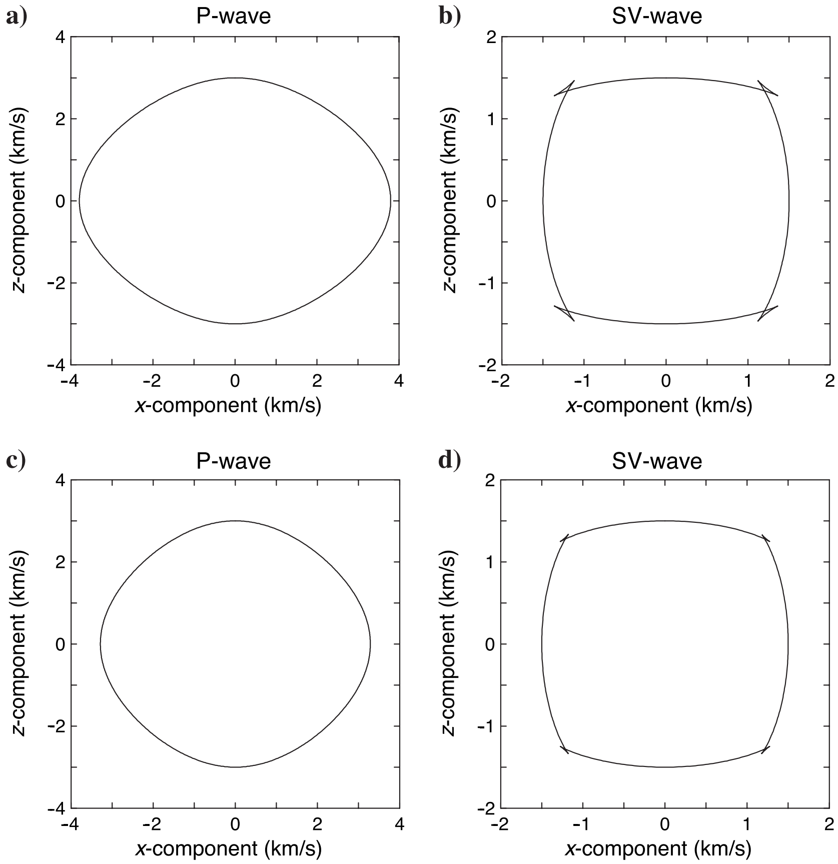

Figure 1. Polar plots of the P- and SV-wave group velocities for models A and B. (a) P-wave velocity in model A, (b) SV-wave velocity in model A, (c) P-wave velocity in model B, and (d) SV-wave velocity in model B. Only parameters of the elastic background medium are considered. For parameters of the models, see Tables 1 and 2.

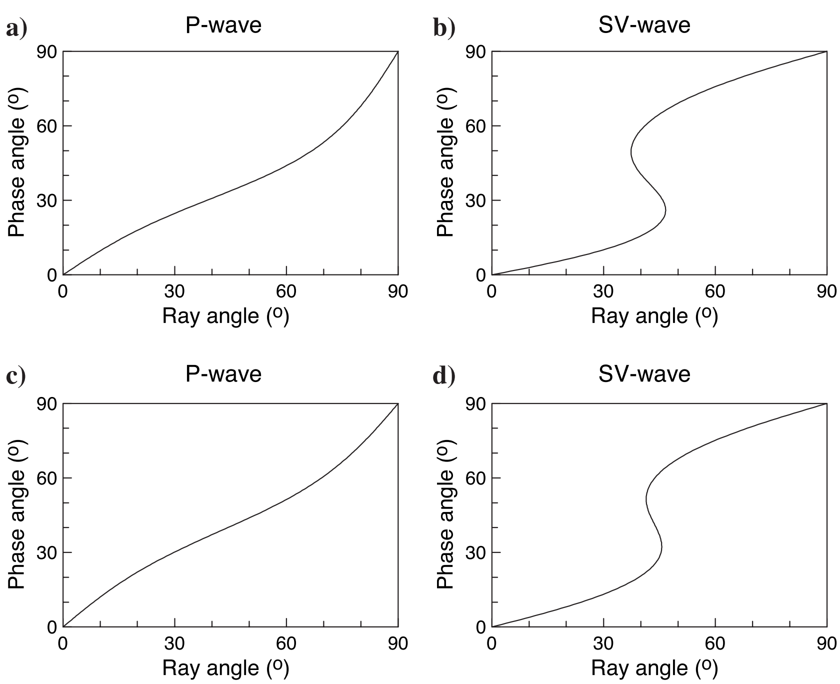

Figure 2. P- and SV-wave phase angles as a function of the ray direction for models A and B. (a) P-wave phase angle in model A, (b) SV-wave phase angle in model A, (c) P-wave phase angle in model B, and (d) SV-wave phase angle in model B. Only parameters of the elastic background medium are considered. For parameters of the models, see Tables 1 and 2.

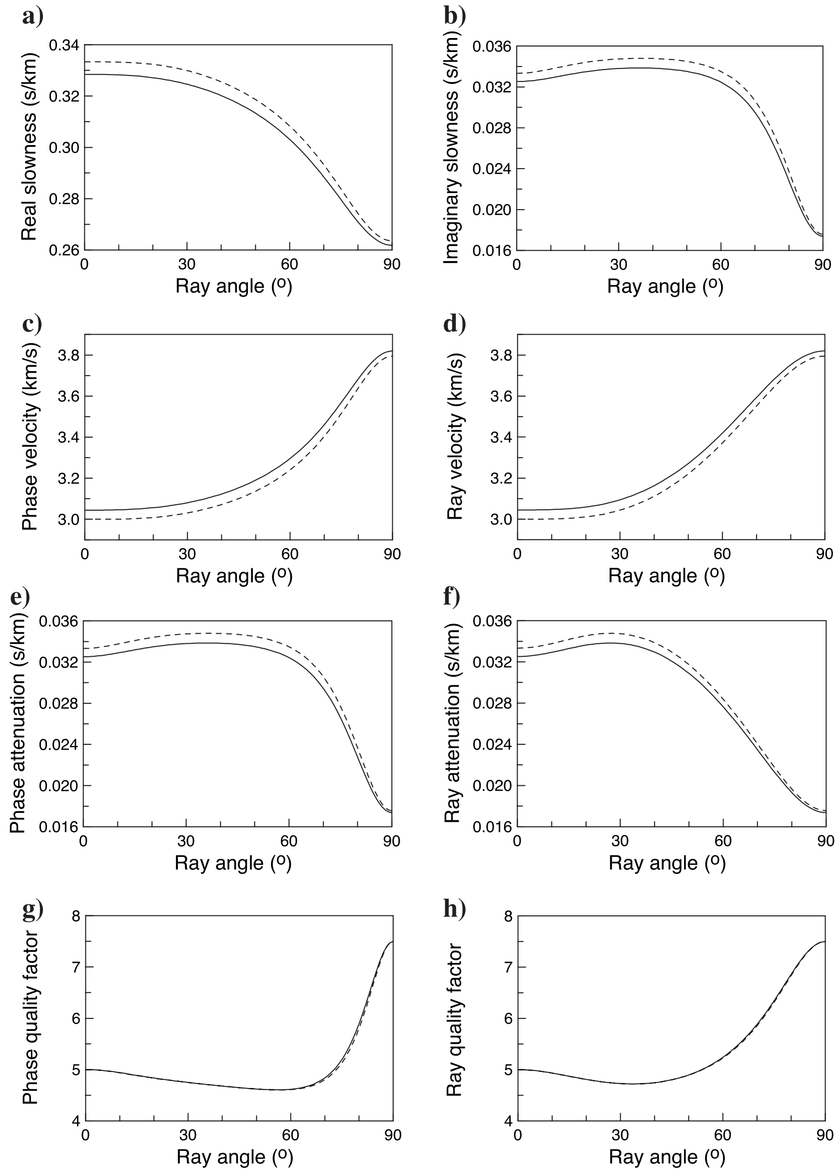

Figure 3. Plots of real and imaginary parts of (a, b) P-wave slowness, (c, d) P-wave phase and ray velocities, (e, f) attenuations, and (g, h) Q-factors in model A1. Solid line — exact formulas; dashed line — first-order perturbations. The ray angle represents the deviation of a ray from the vertical axis.

Table 4. Maximum errors of the first-order perturbations. The error for a particular ray is calculated as $E = 100|U^{\mathrm{exact}} - U^{\mathrm{approx}}|/U^{\mathrm{exact}}$, where U^{exact} and U^{aprox} are the exact and approximate values of the respective quantity. The presented values are the maxima over all rays.

Model	Error $\mathbf{p}^R$ (%)	Error $\mathbf{p}^I$ (%)	Error V^{phase} (%)	Error V^{ray} (%)	Error A^{phase} (%)	Error A^{ray} (%)	Error Q^{phase} (%)	Error Q^{ray} (%)
A1	1.72	4.25	1.63	1.63	2.88	2.79	0.13	0.50
A2	0.44	1.08	0.42	0.42	0.73	0.70	0.10	0.13
A3	0.11	0.29	0.11	0.11	0.19	0.18	0.10	0.04
A4	0.03	0.10	0.03	0.03	0.10	0.05	0.10	0.03
B1	1.66	3.84	1.63	1.63	2.79	2.80	0.10	0.50
B2	0.42	0.97	0.42	0.42	0.70	0.71	0.05	0.13
B3	0.11	0.25	0.11	0.11	0.18	0.18	0.05	0.04
B4	0.03	0.07	0.03	0.03	0.05	0.05	0.05	0.02

Model A1 displays the strongest attenuation among models A1–A4, exhibiting that the perturbation formulas work with the lowest accuracy. For models of weaker attenuation, the accuracy increases. For models A2, A3, and A4, the errors are less than 1%, 0.3%, and 0.1%, respectively (see Table 4). Figure 4 compares the asymptotic solution and the first-order perturbations for model A3, proving that the differences between the solutions are almost invisible. Similar values of errors also are obtained for models B1–B4 (see Table 4).

Figures 5 and 6 compare the correct asymptotics, the first-order perturbations, and the perturbations with improved accuracy for models A1 and B1. The errors are summarized in Table 5. The comparison indicates that the accuracy of the first-order perturbations is improved remarkably by following the procedure described earlier. For

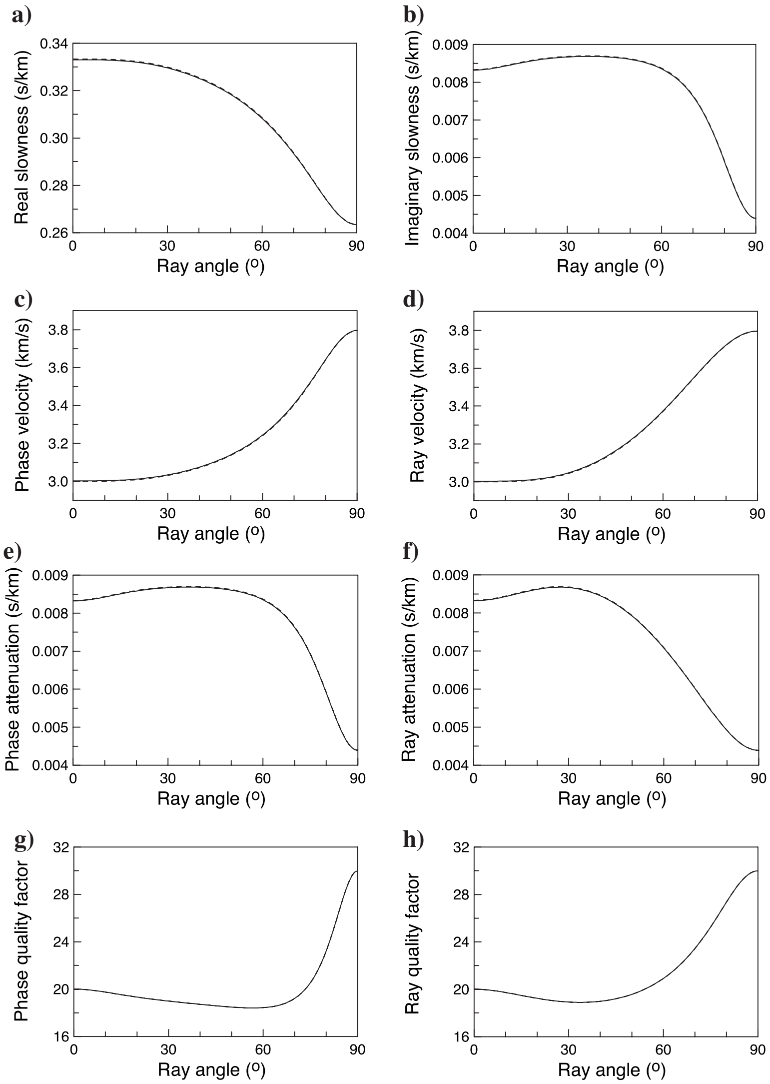

Figure 4. Plots of real and imaginary parts of (a, b) P-wave slowness, (c, d) P-wave phase and ray velocities, (e, f) attenuations, and (g, h) Q-factors in model A3. Solid line — exact formulas; dashed line — first-order perturbations.

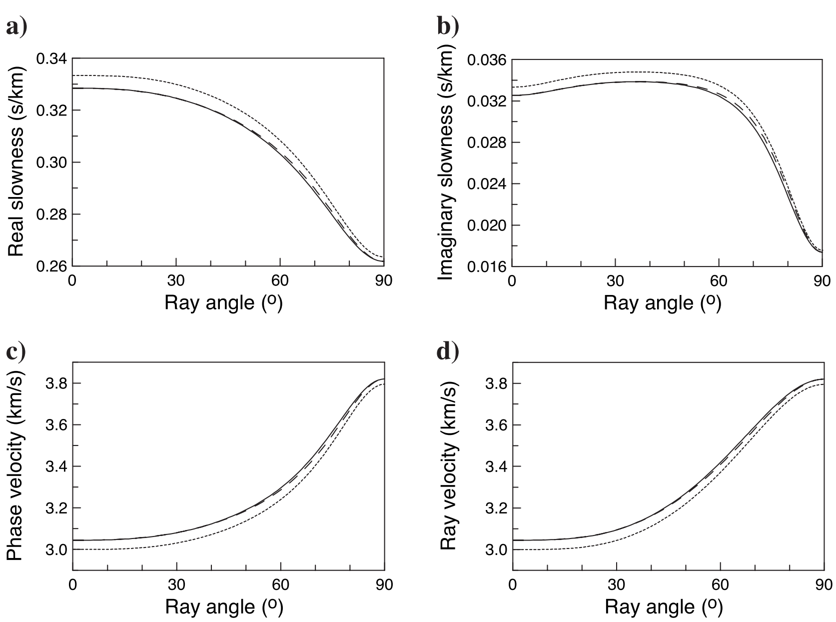

Figure 5. Plots of (a, b) P-wave slowness and (c, d) velocities in model A1. Solid line — exact formulas; dotted line — first-order perturbations; dashed line — perturbations with improved accuracy.

Figure 6. Plots of (a, b) P-wave slowness and (c, d) velocities in model B1. Solid line — exact formulas; dotted line — first-order perturbations; dashed line — perturbations with improved accuracy.

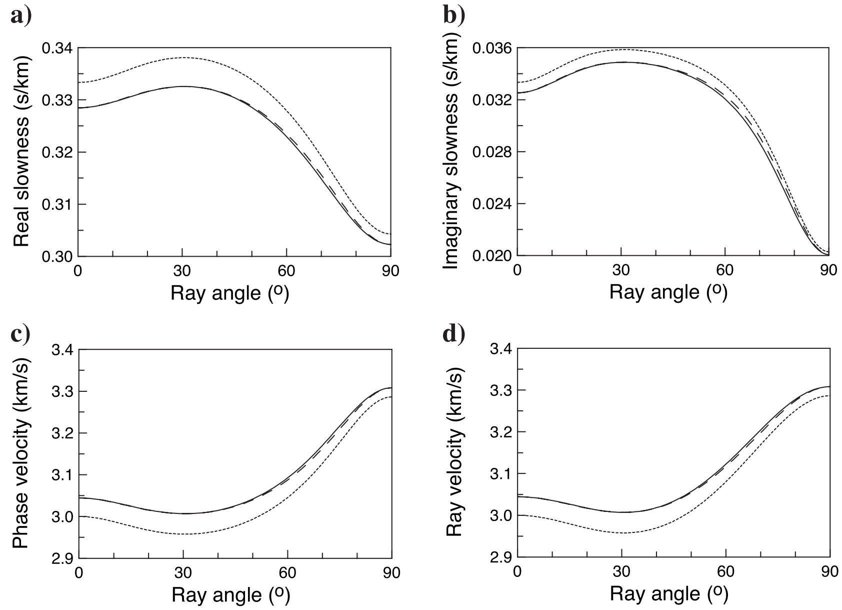

Table 5. Maximum errors of the improved perturbations.

Model	Error $\mathbf{p}^R$ (10^{-1}%)	Error $\mathbf{p}^I$ (10^{-1}%)	Error V^{phase} (10^{-1}%)	Error V^{ray} (10^{-1}%)
A1	4.60	23.62	4.68	3.72
A2	1.25	6.22	1.11	1.02
A3	0.33	1.70	0.29	0.26
A4	0.13	1.17	0.13	0.07
B1	2.36	18.11	2.41	2.16
B2	0.67	4.77	0.72	0.62
B3	0.18	1.27	0.20	0.16
B4	0.05	0.54	0.06	0.04

models A1 and B1, which have the strongest attenuation, the refined formulas yield the errors in the phase and ray velocities less than 0.5%. This documents that the perturbation approach is sufficiently accurate not only for media with weak attenuation but for the whole range of attenuative media, which might be observed in seismology and seismic exploration.

CONCLUSIONS

For calculating asymptotic wave quantities in media with attenuation, it is advantageous to apply the first-order perturbation theory. The elastic medium is considered as a background medium, and the attenuation effects are incorporated as perturbations. Interestingly, some quantities are unaffected by the first-order perturbations being the same for perturbed as well as unperturbed media. This concerns the phase and ray velocities and their directions. The perturbations of the slowness vector, the polarization vector, and the other quantities under study are nonzero.

The most complicated task is to calculate perturbation of the stationary slowness vector $\Delta\mathbf{p}$. Perturbation $\Delta\mathbf{p}$ is calculated by solving a system of three linear equations; it involves inverting the wave metric tensor of the background medium $H^0_{i\ell}$. Equation 38 for calculating $\Delta\mathbf{p}$ is the key result of this paper. Once $\Delta\mathbf{p}$ is evaluated, calculating A^{phase}, Q^{phase}, and D^{phase} is straightforward (see equations 48, 49, and 51). Calculating the ray quantities A^{ray} and Q^{ray} is even simpler than calculating the corresponding phase quantities. The ray quantities can be expressed in terms of medium perturbations $\Delta a_{ijk\ell}$ without the need to calculate $\Delta\mathbf{p}$ (see equations 57 and 59). This finding is important because it can greatly facilitate the inversion for quality parameters from observations of wave attenuation in anisotropic media.

The derived formulas are valid under the following limitations. First, they describe high-frequency wavefields at distances far from the source. Second, the wave must not propagate in directions close to cusp edges on the wave surface because the standard asymptotics fail in these directions. Third, the wave must not propagate near directions associated with singularities (acoustic axes) on the slowness surface because the Christoffel tensor becomes nearly degenerate and the standard perturbation formulas fail.

Numerical modeling documents that the first-order perturbations are highly accurate. They yield a reasonable accuracy even for extremely strong attenuation. The accuracy is 4% or less for media with a Q-factor of about five, and 0.3% or less for media with a Q-factor of 20 or higher. Because most rocks in the earth's crust and in the mantle are characterized by Q-factors higher than 20, the applicability area of the first-order perturbations is very broad, covering applications from global earthquake seismology to seismic exploration. Moreover, the accuracy of perturbations can be enhanced further. If perturbations are used only to calculate the complex direction of the stationary slowness vector and all other computations are performed exactly, the accuracy is remarkably higher. This costs almost no additional computer time and results in errors of the phase and ray velocities less than 0.5% in media with a Q-factor of five and less than 0.03% in media with a Q-factor of 20 or higher.

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