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Weak anisotropy-attenuation parameters

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ABSTRACT

Velocity anisotropy and attenuation in weakly anisotropic and weakly attenuating structures can be treated uniformly using weak anisotropy-attenuation (WAA) parameters. The WAA parameters are constructed in a way analogous to weak anisotropy (WA) parameters designed for weak elastic anisotropy. The WAA parameters generalize WA parameters by incorporating attenuation effects. They can be represented alternatively by one set of complex values or by two sets of real values. Assuming high-frequency waves and using the first-order perturbation theory, all basic wave quantities such as the slowness vector, the polarization vector, propagation velocity, attenuation, and the quality factor are linear functions of WAA parameters. Numerical modeling shows that perturbation equations have different accuracy for different wave quantities. The propagation velocity usually is calculated with high accuracy. However, the attenuation and quality factor can be reproduced with appreciably lower accuracy. This happens mostly when the strength of velocity anisotropy is higher than 10% and attenuation is moderate or weak (Q-factor > 20). In this case, the errors of the attenuation or Q-factor can attain values comparable to the strength of anisotropy or even higher. A simple modification of the equations by including some higher-order perturbations improves accuracy by three to four times.

INTRODUCTION

Anisotropic attenuating media are frequently found in exploration seismics and are studied intensively in the theory of seismic wave propagation (Carcione, 1994, 2000, 2007). Because a general approach valid for wave modeling in anisotropic attenuating media with any strength of anisotropy and attenuation is complicated and demanding computationally (Carcione, 1990; Saenger and Bohlen, 2004), it is advantageous to adopt several simplifying assumptions. First, we often assume the studied waves are of high frequency; second, we assume the medium is weakly anisotropic and/or weakly attenuating. Both conditions are reasonable and frequently met in seismic practice. Imposing these conditions is worthwhile because it allows us to apply (1) a ray theory, designed for propagating high-frequency waves (Červený, 2001), (2) perturbation theory, suitable for solving wave-propagation problems related to weak anisotropy and weak attenuation.

So far, the perturbation theory has been applied mainly to wave-propagation problems in weakly anisotropic elastic media (Thomsen, 1986; Jech and Pšenčík, 1989; Vavryčuk, 1997, 2003; Farra, 2001, 2004; Song et al., 2001; Pšenčík and Vavryčuk, 2002). This medium is introduced as a perturbation of an isotropic elastic background, and anisotropic wave quantities are calculated as perturbations of isotropic wave quantities. The perturbation equations depend linearly on weak anisotropy (WA) parameters, which quantify the elastic anisotropy of the medium (Thomsen, 1986; Mensch and Rasolofosaon, 1997; Rasolofosaon, 2000; Pšenčík and Farra, 2005; Farra and Pšenčík, 2008). A similar approach can be applied to weakly attenuating media where wave quantities in attenuating media are calculated as perturbations of those in nonattenuating media. These approaches can be combined, and the effects of weak anisotropy and weak attenuation can be treated simultaneously and uniformly.

I have developed a perturbation theory applicable to propagating high-frequency waves in weakly anisotropic and weakly attenuating media. All basic wave quantities are expressed in terms of weak anisotropy-attenuation (WAA) parameters, which quantify the velocity and attenuation anisotropy and play a key role in the perturbation equations. They can be defined as complex-valued or real-valued quantities.

Complex WAA parameters were first introduced by Rasolofosaon (2008) and were applied to propagating homogeneous plane waves in weakly anisotropic and weakly attenuating media of arbitrary symmetry. Rasolofosaon (2008) uses the correspondence principle in his derivation and considers an anisotropic viscoelastic reference medium. A complete set of real-valued WAA parameters has not been published yet. Real-valued WAA parameters have a form similar to a linearized version of Thomsen parameters known from studies of elastic and viscoelastic transverse isotropy (Zhu and Tsvankin,

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2006) and orthorhombic anisotropy (Zhu and Tsvankin, 2007).

All previous approaches are based on the assumption of propagation of homogeneous plane waves. Because I deal with high-frequency waves that are generally inhomogeneous, this paper is a step further from homogeneous plane-wave approaches (Carcione, 2000; Chichinina et al., 2006; Červený and Pšenčík, 2005, 2008a, 2008b; Zhu and Tsvankin, 2006, 2007; Rasolofosaon, 2008) toward more realistic wave modeling. This mainly relates to calculating stationary slowness vectors, polarization vectors, and other wave quantities that inherently depend on wave inhomogeneity (see Vavryčuk, 2007a, 2007b). Wave inhomogeneity can be calculated uniquely in the ray theory from an experimental setup (i.e., from source and receiver positions, medium parameters, and boundary conditions) but must be assumed a priori in plane-wave approaches.

This paper is also an extension of previous work (Vavryčuk, 2008) because it assumes the reference background medium is attenuating instead of purely elastic.

PERTURBATION EQUATIONS

A weakly anisotropic and weakly attenuating medium can be viewed as a medium obtained by a small perturbation of an isotropic elastic or viscoelastic reference medium:

$$a_{ijkl} = a^0_{ijkl} + \Delta a_{ijkl}, \quad (1)$$

where a^0_{ijkl} defines the reference medium and Δa_{ijkl} its perturbation. The density-normalized viscoelastic stiffness parameters a^0_{ijkl} can be expressed in terms of the P- and S-wave velocities c^P_0 and c^S_0 in the reference medium:

$$a^0_{ijkl} = ((c^P_0)^2 - 2(c^S_0)^2)\delta_{ij}\delta_{kl} + (c^S_0)^2(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}), \quad (2)$$

where δ_{ij} denotes the Kronecker delta. If the reference medium is elastic, the reference parameters are real and the perturbations are complex,

$$a^0_{ijkl} = a^R_{ijkl}, \quad \Delta a_{ijkl} = \Delta a^R_{ijkl} + i\Delta a^I_{ijkl}, \quad (3)$$

where perturbations Δa^R_{ijkl} and Δa^I_{ijkl} describe weak anisotropy and weak attenuation, respectively. If the reference medium is viscoelastic, the reference parameters and perturbations are complex. To keep the approach as general as possible, the reference medium is considered as viscoelastic.

Using the first-order perturbation theory, we can simplify the equations for the phase and ray quantities derived for homogeneous media of arbitrarily strong anisotropy and attenuation (see Vavryčuk, 2007a, 2007b). The approach is basically the same as presented in Vavryčuk (2008) except we now consider a different reference medium because the reference medium is assumed to be anisotropic elastic in Vavryčuk (2008) and isotropic viscoelastic in this paper. The ray direction is fixed during perturbations. The perturbation of the eigenvalue of the Christoffel tensor $G(\mathbf{n})$,

$$G(\mathbf{n}) = a_{ijkl}n_i n_l g_j g_k = c^2, \quad (4)$$

reads

$$G = G_0 + \Delta G, \quad (5)$$

$$G_0 = c^2_0, \quad \Delta G = \Delta a_{ijkl} n^0_i n^0_l g^0_j g^0_k. \quad (6)$$

The eigenvalue G_0 in the reference medium and the perturbation ΔG are complex valued:

$$G_0 = G^R_0 + iG^I_0, \quad \Delta G = \Delta G^R + i\Delta G^I, \quad (7)$$

$$\Delta G^R = \Delta a^R_{ijkl} n^0_i n^0_l g^0_j g^0_k, \quad (8)$$

$$\Delta G^I = \Delta a^I_{ijkl} n^0_i n^0_l g^0_j g^0_k, \quad (9)$$

where slowness and polarization vectors $\mathbf{n}^0$ and $\mathbf{g}^0$ are real valued and correspond to an isotropic viscoelastic reference medium. For the P-wave, the polarization vector $\mathbf{g}^0$ equals the slowness direction vector $\mathbf{n}^0$. For the S-waves, the polarization vectors $\mathbf{g}^0$ lie in the plane perpendicular to $\mathbf{n}^0$. Their orientation in this plane must be calculated according to perturbation equations designed for degenerate eigenvectors (see Appendix A in Vavryčuk, 2003).

Equations 8 and 9 are valid if perturbations Δa^R_{ijkl} and Δa^I_{ijkl} are mutually comparable and small with respect to reference medium values. Because Δa^I_{ijkl} often is significantly smaller than Δa^R_{ijkl}, equation 9 can appear to have a low accuracy (see Numerical Examples). The inaccuracy is incorporated into equation 9 by identifying slowness direction $\mathbf{n}$ and polarization vector $\mathbf{g}$ in an anisotropic medium with $\mathbf{n}$ and $\mathbf{g}$ in the isotropic reference medium. Thus, the effects of the velocity anisotropy are neglected fully in equation 9. The accuracy is improved if we adopt a modified equation for ΔG^I, expressed as

$$\Delta G^I = \Delta a^I_{ijkl} n^R_i n^R_l g^0_j g^0_k. \quad (10)$$

Even higher accuracy is achieved for ΔG^I, expressed as

$$\Delta G^I = \Delta a^I_{ijkl} n^R_i n^R_l g^R_j g^R_k, \quad (11)$$

where $\mathbf{n}^R$ and $\mathbf{g}^R$ are the real parts of the slowness and polarization vectors in a weakly anisotropic medium, respectively. Similarly, equation 8 for ΔG^R should be modified in a way analogous to equations 10 or 11 if we study the details of a very weakly anisotropic but strongly attenuating medium.

The perturbation of the slowness vector is derived in Appendix A. The slowness vector is homogeneous in the reference medium but generally inhomogeneous in a perturbed medium. However, the inhomogeneity is small, being on the order of the first perturbation. The perturbation of a polarization vector is derived in Appendix B.

If we calculate complex energy velocity v as the magnitude of complex energy velocity vector $\mathbf{v}$,

$$v = \sqrt{v_i v_i}, \quad v_i = a_{ijkl} p_l g_j g_k, \quad (12)$$

and complex phase velocity c from equation 4, we obtain velocities v and c that are equal in the first-order perturbation theory and read

$$v = c = \sqrt{G}. \quad (13)$$

Similarly, other ray and phase quantities (for their definitions, see Vavryčuk, 2007b) are equal in the first-order perturbation theory:

$$V^{phase} = V^{ray}, \quad [Q^{phase}]^{-1} = [Q^{ray}]^{-1}, \quad A^{phase} = A^{ray}. \quad (14)$$

Hence, hereafter I do not distinguish between the ray and phase quantities but speak only of velocity V, quality factor Q, and attenuation A.

The propagation velocity is expressed as

$$V = \sqrt{G^{\mathrm{R}}} = V_0\left(1 + \frac{1}{2}\frac{\Delta G^{\mathrm{R}}}{G_0^{\mathrm{R}}}\right). \qquad (15)$$

This equation follows from the expression $V^2 = V_0^2 + \Delta G^{\mathrm{R}}$, where V_0 corresponds to the velocity in the isotropic elastic part of the reference medium $V_0^2 = G_0^{\mathrm{R}}$. The attenuation and Q-factor read

$$Q^{-1} = Q_V^{-1} - \frac{\Delta G^{\mathrm{I}}}{V^2}, \quad A = A_V - \frac{\Delta G^{\mathrm{I}}}{2V^3}, \qquad (16)$$

or, alternatively,

$$Q^{-1} = Q_V^{-1}\left(1 + \frac{\Delta G^{\mathrm{I}}}{G_0^{\mathrm{I}}}\right), \quad A = A_V\left(1 + \frac{\Delta G^{\mathrm{I}}}{G_0^{\mathrm{I}}}\right), \qquad (17)$$

where

$$G_0 = c_0^2, \quad G_0^{\mathrm{R}} = (c_0^2)^{\mathrm{R}}, \quad G_0^{\mathrm{I}} = (c_0^2)^{\mathrm{I}},$$
$$Q_V^{-1} = -\frac{G_0^{\mathrm{I}}}{V^2},$$
$$A_V = -\frac{G_0^{\mathrm{I}}}{2V^3}. \qquad (18)$$

Equations 16 and 17 follow from equations $Q^{-1} = -G^{\mathrm{I}}/G^{\mathrm{R}}$ and $A = Q^{-1}/2V$ (see equations 44, 51, 54, and 59 in Vavryčuk, 2008). It should be emphasized that Q_V^{-1} and A_V are not quantities describing an isotropic viscoelastic reference medium; they reflect the effects of weak velocity anisotropy being directionally dependent. The dependence on velocity V is acknowledged by using subscript V. Equation 16 holds for viscoelastic and elastic reference media; equation 17 is restricted to the viscoelastic reference medium only (G_0^{I} in the denominator must be nonzero).

Although the phase and ray quantities are equal, the ray and slowness directions differ (see Pšenčík and Vavryčuk, 2002). Ray direction $\mathbf{N}$ is real and fixed and therefore does not change during perturbations $\mathbf{N} = \mathbf{N}^0$. The ray direction $\mathbf{N}$ is equal to the slowness direction $\mathbf{n}^0$ in the isotropic reference medium. However, the slowness direction $\mathbf{n}$ in a perturbed medium deviates from $\mathbf{n}^0$ and $\mathbf{N}$ and is generally complex. The difference between directions $\mathbf{n}$ and $\mathbf{N}$ is on the order of the first perturbation.

A similar observation about the approximate equality of phase and ray attenuations (equation 14) is reported by Behura and Tsvankin (2009), who show that the so-called normalized group attenuation coefficient estimated along seismic rays practically coincides with the phase attenuation coefficient computed for a zero-inhomogeneity angle. However, in strongly anisotropic and attenuating media, the equality of the ray and phase attenuation coefficients can be broken under certain conditions, especially for large inhomogeneity angles (Vavryčuk, 2007b; Behura and Tsvankin, 2009).

WAA PARAMETERS

Instead of using perturbations Δa_{ijkl} in equations for wave quantities, it is convenient to rearrange the equations by introducing the dimensionless WAA parameters. They are constructed very similarly to WA parameters, which are used in weak elastic anisotropy. The WAA parameters can be defined as complex quantities or real quantities. The complex parameters describe a directional variation of the complex energy velocity or, equivalently, of the complex phase velocity and jointly reflect the velocity anisotropy and attenuation. One set of complex WAA parameters can be split into two sets of real WAA parameters, which describe the directional variations of real velocity and real attenuation separately.

Procedure

To construct complex and real WAA parameters, we define dimensionless perturbations $\Delta\varepsilon_{ijkl}$, $\Delta\varepsilon_{ijkl}^V$, and $\Delta\varepsilon_{ijkl}^Q$ as

$$\Delta\varepsilon_{ijkl} = \frac{\Delta a_{ijkl}}{G_0}, \quad \Delta\varepsilon_{ijkl}^V = \frac{\Delta a_{ijkl}^{\mathrm{R}}}{G_0^{\mathrm{R}}}, \quad \Delta\varepsilon_{ijkl}^Q = \frac{\Delta a_{ijkl}^{\mathrm{I}}}{G_0^{\mathrm{I}}}. \qquad (19)$$

Hence,

$$G = G_0(1 + \Delta\varepsilon_{ijkl} n_i^0 n_l^0 g_j^0 g_k^0),$$
$$G^{\mathrm{R}} = G_0^{\mathrm{R}}(1 + \Delta\varepsilon_{ijkl}^V n_i^0 n_l^0 g_j^0 g_k^0),$$
$$G^{\mathrm{I}} = G_0^{\mathrm{I}}(1 + \Delta\varepsilon_{ijkl}^Q n_i^0 n_l^0 g_j^0 g_k^0). \qquad (20)$$

Using the notation

$$\Delta\varepsilon = \Delta\varepsilon_{ijkl} n_i^0 n_l^0 g_j^0 g_k^0,$$
$$\Delta\varepsilon^V = \Delta\varepsilon_{ijkl}^V n_i^0 n_l^0 g_j^0 g_k^0,$$
$$\Delta\varepsilon^Q = \Delta\varepsilon_{ijkl}^Q n_i^0 n_l^0 g_j^0 g_k^0, \qquad (21)$$

the equations for the eigenvalue of the Christoffel tensor, phase velocity, Q-factor, and attenuation read

$$G = G_0(1 + \Delta\varepsilon), \quad V = V_0\left(1 + \frac{1}{2}\Delta\varepsilon^V\right),$$
$$Q^{-1} = Q_V^{-1}(1 + \Delta\varepsilon^Q), \quad A = A_V(1 + \Delta\varepsilon^Q), \qquad (22)$$

where

$$G_0 = c_0^2, \quad V_0 = \sqrt{(c_0^2)^{\mathrm{R}}}, \quad Q_V^{-1} = -\frac{G_0^{\mathrm{I}}}{V^2}, \quad A_V = -\frac{G_0^{\mathrm{I}}}{2V^3}. \qquad (23)$$

Quantities G_0 and V_0 describe the isotropic reference medium and are directionally independent. Quality factor Q_V and attenuation A_V are directionally dependent.

Definition of complex WAA parameters

To keep the notation consistent with WA parameters defined previously by Farra and Pšenčík (2008; their equation A1), the dimensionless perturbations $\Delta\varepsilon_{ijkl}$, are expressed in Voigt notation and slightly rearranged. Hence, complex WAA parameters are defined ultimately as

$$\varepsilon_x = \frac{a_{11} - G_0^{\mathrm{P}}}{2G_0^{\mathrm{P}}}, \quad \varepsilon_y = \frac{a_{22} - G_0^{\mathrm{P}}}{2G_0^{\mathrm{P}}}, \quad \varepsilon_z = \frac{a_{33} - G_0^{\mathrm{P}}}{2G_0^{\mathrm{P}}},$$

$$\delta_x = \frac{a_{23} + 2a_{44} - G_0^{\mathrm{P}}}{G_0^{\mathrm{P}}}, \quad \delta_y = \frac{a_{13} + 2a_{55} - G_0^{\mathrm{P}}}{G_0^{\mathrm{P}}},$$

$$\delta_z = \frac{a_{12} + 2a_{66} - G_0^{\mathrm{P}}}{G_0^{\mathrm{P}}},$$

$$\gamma_x = \frac{a_{44} - G_0^{\mathrm{S}}}{2G_0^{\mathrm{S}}}, \quad \gamma_y = \frac{a_{55} - G_0^{\mathrm{S}}}{2G_0^{\mathrm{S}}}, \quad \gamma_z = \frac{a_{66} - G_0^{\mathrm{S}}}{2G_0^{\mathrm{S}}},$$

$$\chi_x = \frac{a_{14} + 2a_{56}}{G_0^{\mathrm{P}}}, \quad \chi_y = \frac{a_{25} + 2a_{46}}{G_0^{\mathrm{P}}},$$

$$\chi_z = \frac{a_{36} + 2a_{45}}{G_0^{\mathrm{P}}},$$

$$\varepsilon_{15} = \frac{a_{15}}{G_0^{\mathrm{P}}}, \quad \varepsilon_{16} = \frac{a_{16}}{G_0^{\mathrm{P}}}, \quad \varepsilon_{24} = \frac{a_{24}}{G_0^{\mathrm{P}}}, \quad \varepsilon_{26} = \frac{a_{26}}{G_0^{\mathrm{P}}},$$

$$\varepsilon_{34} = \frac{a_{34}}{G_0^{\mathrm{P}}}, \quad \varepsilon_{35} = \frac{a_{35}}{G_0^{\mathrm{P}}},$$

$$\varepsilon_{46} = \frac{a_{46}}{G_0^{\mathrm{S}}}, \quad \varepsilon_{56} = \frac{a_{56}}{G_0^{\mathrm{S}}}, \quad \varepsilon_{45} = \frac{a_{45}}{G_0^{\mathrm{S}}}, \tag{24}$$

where a_{ij} are complex viscoelastic parameters in the Voigt notation and where G_0^{P} and G_0^{S} are complex eigenvalues of the Christoffel tensor in the isotropic viscoelastic reference medium corresponding to the P- and S-waves. They can be calculated from real P- and S-wave velocities α and β and quality factors Q_0^{P} and Q_0^{S} as follows:

$$G_0^{\mathrm{P}} = \alpha^2\left(1 - \frac{i}{Q_0^{\mathrm{P}}}\right), \quad G_0^{\mathrm{S}} = \beta^2\left(1 - \frac{i}{Q_0^{\mathrm{S}}}\right). \tag{25}$$

Definition of real WAA parameters

If we separate the effects of velocity anisotropy and attenuation, we obtain two sets of real WAA parameters: one for the velocity anisotropy (with superscript V) and one for the attenuation anisotropy (with superscript Q). Again, perturbations $\Delta\varepsilon_{ijkl}^{V}$ and $\Delta\varepsilon_{ijkl}^{Q}$ are rearranged in a way similar to equation 24. For the velocity anisotropy parameters, we obtain

$$\varepsilon_x^V = \frac{a_{11}^{\mathrm{R}} - \alpha^2}{2\alpha^2}, \quad \varepsilon_y^V = \frac{a_{22}^{\mathrm{R}} - \alpha^2}{2\alpha^2}, \quad \varepsilon_z^V = \frac{a_{33}^{\mathrm{R}} - \alpha^2}{2\alpha^2},$$

$$\delta_x^V = \frac{a_{23}^{\mathrm{R}} + 2a_{44}^{\mathrm{R}} - \alpha^2}{\alpha^2}, \quad \delta_y^V = \frac{a_{13}^{\mathrm{R}} + 2a_{55}^{\mathrm{R}} - \alpha^2}{\alpha^2},$$

$$\delta_z^V = \frac{a_{12}^{\mathrm{R}} + 2a_{66}^{\mathrm{R}} - \alpha^2}{\alpha^2},$$

$$\gamma_x^V = \frac{a_{44}^{\mathrm{R}} - \beta^2}{2\beta^2}, \quad \gamma_y^V = \frac{a_{55}^{\mathrm{R}} - \beta^2}{2\beta^2}, \quad \gamma_z^V = \frac{a_{66}^{\mathrm{R}} - \beta^2}{2\beta^2},$$

$$\chi_x^V = \frac{a_{14}^{\mathrm{R}} + 2a_{56}^{\mathrm{R}}}{\alpha^2}, \quad \chi_y^V = \frac{a_{25}^{\mathrm{R}} + 2a_{46}^{\mathrm{R}}}{\alpha^2},$$

$$\chi_z^V = \frac{a_{36}^{\mathrm{R}} + 2a_{45}^{\mathrm{R}}}{\alpha^2},$$

$$\varepsilon_{15}^V = \frac{a_{15}^{\mathrm{R}}}{\alpha^2}, \quad \varepsilon_{16}^V = \frac{a_{16}^{\mathrm{R}}}{\alpha^2}, \quad \varepsilon_{24}^V = \frac{a_{24}^{\mathrm{R}}}{\alpha^2}, \quad \varepsilon_{26}^V = \frac{a_{26}^{\mathrm{R}}}{\alpha^2},$$

$$\varepsilon_{34}^V = \frac{a_{34}^{\mathrm{R}}}{\alpha^2}, \quad \varepsilon_{35}^V = \frac{a_{35}^{\mathrm{R}}}{\alpha^2},$$

$$\varepsilon_{46}^V = \frac{a_{46}^{\mathrm{R}}}{\beta^2}, \quad \varepsilon_{56}^V = \frac{a_{56}^{\mathrm{R}}}{\beta^2}, \quad \varepsilon_{45}^V = \frac{a_{45}^{\mathrm{R}}}{\beta^2}. \tag{26}$$

The attenuation anisotropy parameters are defined analogously as the velocity anisotropy parameters but in terms of a_{ij}^{I}, Q_0^{P}, and Q_0^{S},

$$\varepsilon_x^Q = -\frac{a_{11}^{\mathrm{I}}Q_0^{\mathrm{P}} + \alpha^2}{2\alpha^2}, \quad \varepsilon_y^Q = -\frac{a_{22}^{\mathrm{I}}Q_0^{\mathrm{P}} + \alpha^2}{2\alpha^2},$$

$$\varepsilon_z^Q = -\frac{a_{33}^{\mathrm{I}}Q_0^{\mathrm{P}} + \alpha^2}{2\alpha^2},$$

$$\delta_x^Q = -\frac{(a_{23}^{\mathrm{I}} + 2a_{44}^{\mathrm{I}})Q_0^{\mathrm{P}} + \alpha^2}{\alpha^2},$$

$$\delta_y^Q = -\frac{(a_{13}^{\mathrm{I}} + 2a_{55}^{\mathrm{I}})Q_0^{\mathrm{P}} + \alpha^2}{\alpha^2},$$

$$\delta_z^Q = -\frac{(a_{12}^{\mathrm{I}} + 2a_{66}^{\mathrm{I}})Q_0^{\mathrm{P}} + \alpha^2}{\alpha^2},$$

$$\gamma_x^Q = -\frac{a_{44}^{\mathrm{I}}Q_0^{\mathrm{S}} + \beta^2}{2\beta^2},$$

$$\gamma_y^Q = -\frac{a_{55}^{\mathrm{I}}Q_0^{\mathrm{S}} + \beta^2}{2\beta^2},$$

$$\gamma_z^Q = -\frac{a_{66}^{\mathrm{I}}Q_0^{\mathrm{S}} + \beta^2}{2\beta^2},$$

$$\chi_x^Q = -\frac{a_{14}^{\mathrm{I}} + 2a_{56}^{\mathrm{I}}}{\alpha^2}Q_0^{\mathrm{P}}, \quad \chi_y^Q = -\frac{a_{25}^{\mathrm{I}} + 2a_{46}^{\mathrm{I}}}{\alpha^2}Q_0^{\mathrm{P}},$$

$$\chi_z^Q = -\frac{a_{36}^{\mathrm{I}} + 2a_{45}^{\mathrm{I}}}{\alpha^2}Q_0^{\mathrm{P}},$$

$$\varepsilon_{15}^Q = -\frac{a_{15}^{\mathrm{I}}}{\alpha^2}Q_0^{\mathrm{P}}, \quad \varepsilon_{16}^Q = -\frac{a_{16}^{\mathrm{I}}}{\alpha^2}Q_0^{\mathrm{P}},$$

$$\varepsilon_{24}^Q = -\frac{a_{24}^{\mathrm{I}}}{\alpha^2}Q_0^{\mathrm{P}}, \quad \varepsilon_{26}^Q = -\frac{a_{26}^{\mathrm{I}}}{\alpha^2}Q_0^{\mathrm{P}},$$

$$\varepsilon_{34}^Q = -\frac{a_{34}^{\mathrm{I}}}{\alpha^2}Q_0^{\mathrm{P}}, \quad \varepsilon_{35}^Q = -\frac{a_{35}^{\mathrm{I}}}{\alpha^2}Q_0^{\mathrm{P}},$$

$$\varepsilon_{46}^Q = -\frac{a_{46}^{\mathrm{I}}}{\beta^2}Q_0^{\mathrm{S}}, \quad \varepsilon_{56}^Q = -\frac{a_{56}^{\mathrm{I}}}{\beta^2}Q_0^{\mathrm{S}},$$

$$\varepsilon_{45}^Q = -\frac{a_{45}^{\mathrm{I}}}{\beta^2}Q_0^{\mathrm{S}}. \tag{27}$$

The two sets of real WAA parameters do not coincide with the real and imaginary parts of the one set of complex WAA parameters. This is because complex WAA parameters do not separate the effects of velocity and attenuation anisotropy (see equation 19). For example, the real parts of the complex WAA parameters are affected not only by the velocity anisotropy but also by the attenuation of the reference medium. On the other hand, the two sets of real WAA parameters strictly separate the effects of the velocity and attenuation anisotropy. The velocity anisotropy parameters are not affected by attenuation, and attenuation anisotropy parameters are independent of the elastic anisotropy or the elastic properties of the reference medium. The reader is reminded that the equations for the attenuation anisotropy parameters fail for the elastic reference medium. In this case, only the approach with complex WAA parameters is applicable.

P-WAVE IN TI MEDIA

In this section, the derived equations are specified for the P-wave propagating in a transversely isotropic medium with a vertical axis of symmetry (VTI medium). The medium is described by the following parameters in Voigt notation: $a_{11}, a_{22} = a_{11}, a_{33}, a_{44}, a_{55} = a_{44}, a_{66}, a_{13}, a_{23} = a_{13}$, and $a_{12} = a_{11} - 2a_{66}$. All other parameters are zero. The parameters a_{ij} are complex valued. The velocity anisotropy and attenuation are assumed to be weak. The wave quantities are studied in the x_1-x_3 plane. Perturbations for the SV-wave can be found analogously to the P-wave, and the SH-wave quantities can readily be calculated exactly in the VTI medium.

Equations using perturbations of viscoelastic parameters

The complex and real velocities, quality factors, and attenuations for the P-wave are expressed by the following equations:

$$c^2 = c_0^2 + \Delta G, \quad V = V_0\left(1 + \frac{1}{2}\frac{\Delta G^{\mathrm{R}}}{V_0^2}\right),$$
$$Q^{-1} = Q_V^{-1} - \frac{\Delta G^{\mathrm{I}}}{V^2}, \quad A = A_V - \frac{\Delta G^{\mathrm{I}}}{2V^3}, \qquad (28)$$

where

$$\Delta G = \Delta a_{11} N_1^4 + \Delta a_{33} N_3^4 + 2(\Delta a_{13} + 2\Delta a_{44}) N_1^2 N_3^2,$$
$$\Delta G^{\mathrm{R}} = \Delta a_{11}^{\mathrm{R}} N_1^4 + \Delta a_{33}^{\mathrm{R}} N_3^4 + 2(\Delta a_{13}^{\mathrm{R}} + 2\Delta a_{44}^{\mathrm{R}}) N_1^2 N_3^2,$$
$$\Delta G^{\mathrm{I}} = \Delta a_{11}^{\mathrm{I}} N_1^4 + \Delta a_{33}^{\mathrm{I}} N_3^4 + 2(\Delta a_{13}^{\mathrm{I}} + 2\Delta a_{44}^{\mathrm{I}}) N_1^2 N_3^2. \qquad (29)$$

The reference quantities in equation 28 read

$$c_0 = \alpha\sqrt{1 - \frac{i}{Q_0^{\mathrm{P}}}}, \quad V_0 = \alpha,$$
$$Q_V^{-1} = \frac{\alpha^2}{V^2}\frac{1}{Q_0^{\mathrm{P}}}, \quad A_V = \frac{\alpha^2}{2V^3}\frac{1}{Q_0^{\mathrm{P}}}. \qquad (30)$$

Vector $\mathbf{N}$ is the real ray direction vector, $\mathbf{N} = (\sin\theta, 0, \cos\theta)^T$, quantities α and Q_0^{P} are the real P-wave velocity and Q-factor in the isotropic viscoelastic reference medium, and angle θ defines the deviation of a ray from the symmetry axis.

Equations using WAA parameters

The complex and real velocities, Q-factors, and attenuations for the P-wave are expressed in terms of WAA parameters by the following equations:

$$c^2 = c_0^2(1 + \Delta\varepsilon), \quad V = V_0\left(1 + \frac{1}{2}\Delta\varepsilon^V\right),$$
$$Q^{-1} = Q_V^{-1}(1 + \Delta\varepsilon^Q), \quad A = A_V(1 + \Delta\varepsilon^Q). \qquad (31)$$

Perturbations $\Delta\varepsilon$, $\Delta\varepsilon^V$, and $\Delta\varepsilon^Q$ in equation 31 read

$$\Delta\varepsilon = 2(\varepsilon_x N_1^4 + \varepsilon_z N_3^4 + \delta_x N_1^2 N_3^2),$$
$$\Delta\varepsilon^V = 2(\varepsilon_x^V N_1^4 + \varepsilon_z^V N_3^4 + \delta_x^V N_1^2 N_3^2),$$
$$\Delta\varepsilon^Q = 2(\varepsilon_x^Q N_1^4 + \varepsilon_z^Q N_3^4 + \delta_x^Q N_1^2 N_3^2), \qquad (32)$$

where $N_1 = \sin\theta$ and $N_3 = \cos\theta$. The reference quantities are defined in equation 30, and the WAA parameters are defined in equations 24, 26, and 27.

Equations with improved accuracy

The accuracy of first-order perturbations for A and Q can be improved by incorporating higher-order perturbations. This can be done when treating the slowness vector in a more accurate way than in standard equations. So far, the slowness direction $\mathbf{n}$ was identified simply with ray direction $\mathbf{N}$ in equations 29 and 32. This approximation works well for very weak anisotropy. The stronger the anisotropy, the lower the accuracy of this approximation. Hence, instead of using slowness direction $\mathbf{n}^0 = \mathbf{N}$ in equation 9, we can use the linearized $\mathbf{n}^{\mathrm{R}}$:

$$\mathbf{n}^{\mathrm{R}} = \mathbf{n}^0 + \Delta\mathbf{n}^{\mathrm{R}} = \mathbf{N} + \Delta\mathbf{n}^{\mathrm{R}}. \qquad (33)$$

The perturbation equation for $\Delta\mathbf{n}^{\mathrm{R}}$ is derived in Appendix A for anisotropy of arbitrary symmetry and in Appendix C for transverse isotropy. Hence, in TI media we obtain for the P-wave,

$$\Delta n_1^{\mathrm{R}} = -2\frac{N_1}{\alpha^2}[A_1^{\mathrm{R}} N_3^4 + A_2^{\mathrm{R}} N_3^2],$$
$$\Delta n_3^{\mathrm{R}} = -2\frac{N_3}{\alpha^2}[A_1^{\mathrm{R}} N_3^4 + (A_2^{\mathrm{R}} - A_1^{\mathrm{R}}) N_3^2 - A_2^{\mathrm{R}}]. \qquad (34)$$

Constants A_1^{R} and A_2^{R} are expressed in terms of perturbations $\Delta a_{ijkl}^{\mathrm{R}}$ as

$$A_1^{\mathrm{R}} = -\Delta a_{11}^{\mathrm{R}} + 2\Delta a_{13}^{\mathrm{R}} - \Delta a_{33}^{\mathrm{R}} + 4\Delta a_{44}^{\mathrm{R}},$$
$$A_2^{\mathrm{R}} = \Delta a_{11}^{\mathrm{R}} - \Delta a_{13}^{\mathrm{R}} - 2\Delta a_{44}^{\mathrm{R}} \qquad (35)$$

and in terms of WAA parameters such as

$$A_1^{\mathrm{R}} = 2\alpha^2(\delta_x^V - \varepsilon_x^V - \varepsilon_z^V), \quad A_2^{\mathrm{R}} = \alpha^2(-\delta_x^V + 2\varepsilon_x^V). \qquad (36)$$

Because we correct the slowness direction but not the polarization vectors in equation 10, substituting ray direction $\mathbf{N}$ by the corrected slowness direction $\mathbf{n}$ in equations 29 and 32 will read as follows:

$$\mathbf{n} = \mathbf{n}^0 + \frac{1}{2}\Delta\mathbf{n}^{\mathrm{R}} = \mathbf{N} + \frac{1}{2}\Delta\mathbf{n}^{\mathrm{R}}. \qquad (37)$$

Hence, the corrected equation 32 reads

$$\Delta\varepsilon = 2(\varepsilon_x n_1^4 + \varepsilon_z n_3^4 + \delta_x n_1^2 n_3^2),$$
$$\Delta\varepsilon^V = 2(\varepsilon_x^V n_1^4 + \varepsilon_z^V n_3^4 + \delta_x^V n_1^2 n_3^2),$$

$$\Delta\varepsilon^{Q} = 2(\varepsilon_x^Q n_1^4 + \varepsilon_z^Q n_3^4 + \delta_x^Q n_1^2 n_3^2), \quad (38)$$

where

$$n_1 = N_1\{1 - (\delta_x^V - 2\varepsilon_z^V)N_3^2 - 2(\varepsilon_x^V + \varepsilon_z^V - \delta_x^V)N_1^2 N_3^2\},$$
$$n_3 = N_3\{1 + (\delta_x^V - 2\varepsilon_z^V)N_1^2 + 2(\varepsilon_x^V + \varepsilon_z^V - \delta_x^V)N_1^4\}, \quad (39)$$

and **n** in equation 39 is further normalized to be of a unit length before being inserted into equation 38.

NUMERICAL EXAMPLES

In this section, I demonstrate the accuracy of the perturbation equations using numerical examples for the P-wave in homogeneous VTI media. I adopt four viscoelastic models with two strengths of anisotropy and two levels of attenuation. The models are denoted as models A2, A4, B2, and B4, taken from Vavryčuk (2008). The anisotropy strength (i.e., the magnitude of the directional velocity variation) is 23% for models A2 and A4 and 10% for models B2 and B4. The average Q-factors are about 10 for models A2 and B2 and 40 for models A4 and B4. The Q-factor anisotropy is 46% for all four models (see Table 3 in Vavryčuk, 2008). The models with an anisotropy strength of 23% cannot be considered weakly anisotropic. Here, they are used to illustrate how the accuracy of the perturbation equations deteriorates in this case. The viscoelastic parameters of the models are summarized in Table 1. For detailed information on these models, see Vavryčuk (2008).

Table 1. Viscoelastic parameters. The two-index Voigt notation is used for density-normalized elastic parameters and for quality parameters. Parameters a_{66}^R and Q_{66} are not listed because the P-wave is not sensitive to them.

	Elastic parameters				Attenuation parameters			
Model	a_{11}^R (km²/s²)	a_{13}^R (km²/s²)	a_{33}^R (km²/s²)	a_{44}^R (km²/s²)	Q_{11}	Q_{13}	Q_{33}	Q_{44}
A2	14.4	4.50	9.00	2.25	15	8	10	8
A4	14.4	4.50	9.00	2.25	60	32	40	32
B2	10.8	3.53	9.00	2.25	15	8	10	8
B4	10.8	3.35	9.00	2.25	60	32	40	32

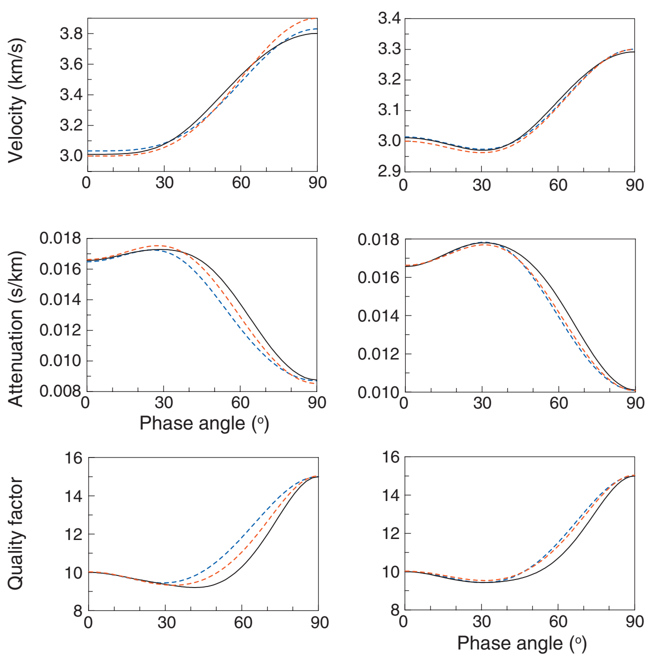

Figure 1. Exact and approximate velocities, attenuations, and Q-factors in models A2 (left-hand plots) and B2 (right-hand plots). Black solid lines show exact phase quantities. Blue dashed lines show approximate quantities calculated using equations 31 and 32. Red dashed lines show approximate quantities calculated using equations 41 and 42. The phase angle denotes the deviation of the real part of the complex slowness vector from the symmetry axis.

Figure 1 shows the directional variations of the exact and approximate velocities, attenuations A, and Q-factors for models A2 (left-hand plots) and B2 (right-hand plots), respectively. Figure 2 shows the same quantities for models A4 and B4. The angles range from 0° to 90°. The exact phase quantities (black solid line) are calculated according to equations 40, 41, and 44 of Vavryčuk (2008). The exact stationary slowness vector is calculated by a procedure described in Vavryčuk (2007b). The approximate velocities, attenuations A, and Q-factors are calculated using equations 31 and 32 (blue dashed line). The reference quantities needed in the approximate equations are listed in Table 2.

In the approximations, I do not distinguish between the ray and phase quantities because they are identical in the first-order perturbation theory. The figures show that the highest accuracy is achieved for the velocity having errors less than 2% for A models and less than 1% for B models. This result is satisfactory, considering that the strength of

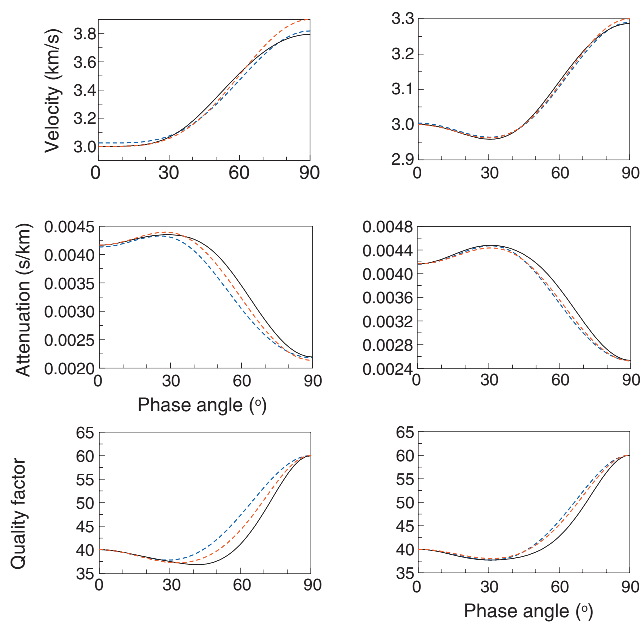

Figure 2. Exact and approximate velocities, attenuations, and Q-factors in models A4 (left-hand plots) and B4 (right-hand plots). For details, see Figure 1.

the velocity anisotropy is 23% for A models and 10% for B models. However, the accuracies of A and Q are considerably lower — approximately 12–15% for models A2 and A4 and 7–8% for models B2 and B4 (see Tables 3 and 4).

To assess the effectiveness and accuracy of perturbation equations 31 and 32, Figures 1 and 2 show the approximate velocities, attenuations, and Q-factors calculated using alternative equations derived for the P-wave propagating in TI media (red dashed lines) and exploiting the Thomsen parameters (Thomsen, 1986; Tsvankin, 2005; Zhu and Tsvankin, 2006),

$$V^{\mathrm{Th}} = V_0^{\mathrm{Th}}(1 + \delta N_1^2 N_3^2 + \varepsilon N_1^4), \qquad (40)$$

$$A^{\mathrm{Th}} = \frac{A_0^{\mathrm{Th}}}{V^{\mathrm{Th}}}(1 + \delta_Q N_1^2 N_3^2 + \varepsilon_Q N_1^4), \qquad (41)$$

$$Q^{\mathrm{Th}} = \frac{1}{2A^{\mathrm{Th}}V^{\mathrm{Th}}}, \qquad (42)$$

where superscript Th denotes Thomsen's quantities. V_0^{Th} is the vertical velocity in the elastic reference VTI medium and ε and δ are Thomsen parameters (equations 8a and 17 in Thomsen, 1986), A_0^{Th} is the reference attenuation (Zhu and Tsvankin, 2006; their equation 22), and ε_Q and δ_Q are the attenuation parameters (Zhu and Tsvankin, 2006; their equations 28 and 31). Because my definition of A differs slightly from that in Zhu and Tsvankin (2006), equation 41 is not identical to the original equation 36 of Zhu and Tsvankin (2006). The values of the Thomsen parameters used in numerical modeling are summarized in Table 2 of Vavryčuk (2008).

Figures 1 and 2 show that the accuracy of equations 41 and 42 for attenuation and Q-factor in models A2 and A4 is almost two times higher than first-order perturbations 31 and 32. For models B2 and B4, accuracy is approximately the same for both approaches. This demonstrates that equations 41 and 42 are preferable in models with stronger velocity anisotropy. The higher accuracy is achieved because δ_Q in equations 41 and 42 depends not only on the attenuation of the medium but also on its velocity anisotropy. This property is lost in real-valued WAA parameters (equations 26 and 27), in which the effects of the velocity anisotropy and attenuation anisotropy are separated fully. Therefore, equations 41 and 42 can be viewed as perturbation equations that incorporate some higher-order terms.

Interestingly, the accuracy of approximate A and Q in Figures 1 and 2 does not depend on the strength of attenuation, even though one would expect the perturbations to work better for less-attenuating media (models A4 and B4). This observation is reported by Zhu and Tsvankin (2006) and is explained by the fact that the accuracy of attenuation is not only affected by the strength of attenuation but also by the strength of the velocity anisotropy. The velocity anisotropy and attenuation are described by perturbations, and their effects cannot be separated easily. Hence, the accuracy of A and Q in the models studied is affected more by the strength of anisotropy than by the strength of attenuation.

If we use the modified perturbation equation 10 for ΔG^{I}, the accuracy of A and Q improves. This is indicated in Figure 3 for models A2 and B2 and is summarized in Tables 3 and 4. The figure and tables also show errors of approximate equations 41 and 42 derived by Zhu and Tsvankin (2006). Both approaches incorporate some of the higher-order perturbations and yield higher accuracy than first-order perturbations. The accuracy of equations 41 and 42 is almost two times higher than standard first-order perturbations. The accuracy of equations 31, 38, and 39 is almost three to four times higher than standard first-order perturbations. Obviously, more complicated approximations (e.g., Zhu and Tsvankin, 2006; their equation 19) can yield even higher accuracy.

Table 2. Values of the isotropic viscoelastic reference medium.

Model	α (km/s)	β (km/s)	Q_0^{P}	Q_0^{S}
A2	3.40	1.50	10.5	8.0
A4	3.40	1.50	42.0	32.0
B2	3.15	1.50	10.5	8.0
B4	3.15	1.50	42.0	32.0

Table 3. Maximum errors of the perturbations of the attenuation. The error for a particular ray is calculated as $E = 100|U^{\mathrm{exact}} - U^{\mathrm{approx}}|/U^{\mathrm{exact}}$, where U^{exact} and U^{approx} are the exact and approximate values of the respective quantity. The presented values are the maxima over all rays. ZT is from equations 41 and 42. V1 is from equations 31 and 32. V2 is from equations 31, 38, and 39.

	Error-ZT		Error-V1		Error-V2	
Model	A^{phase} (%)	A^{ray} (%)	A^{phase} (%)	A^{ray} (%)	A^{phase} (%)	A^{ray} (%)
A2	6.3	10.7	11.6	14.7	2.9	3.1
A4	6.6	11.0	11.8	14.9	2.7	3.3
B2	5.0	6.3	6.8	8.0	0.4	1.6
B4	5.3	6.6	6.9	8.3	0.5	1.8

Table 4. Maximum errors of the perturbations of the Q-factor. The error for a particular ray is calculated as $E = 100|U^{\mathrm{exact}} - U^{\mathrm{approx}}|/U^{\mathrm{exact}}$, where U^{exact} and U^{approx} are the exact and approximate values of the respective quantity. The presented values are the maxima over all rays. ZT is from equations 41 and 42. V1 is from equations 31 and 32. V2 is from equations 31, 38, and 39.

	Error-ZT		Error-V1		Error-V2	
Model	Q^{phase} (%)	Q^{ray} (%)	Q^{phase} (%)	Q^{ray} (%)	Q^{phase} (%)	Q^{ray} (%)
A2	7.6	8.0	15.0	14.2	5.1	4.0
A4	7.3	7.8	14.9	14.3	5.1	4.1
B2	6.1	6.1	8.0	8.0	2.0	2.1
B4	5.9	6.0	8.0	8.1	2.0	2.2

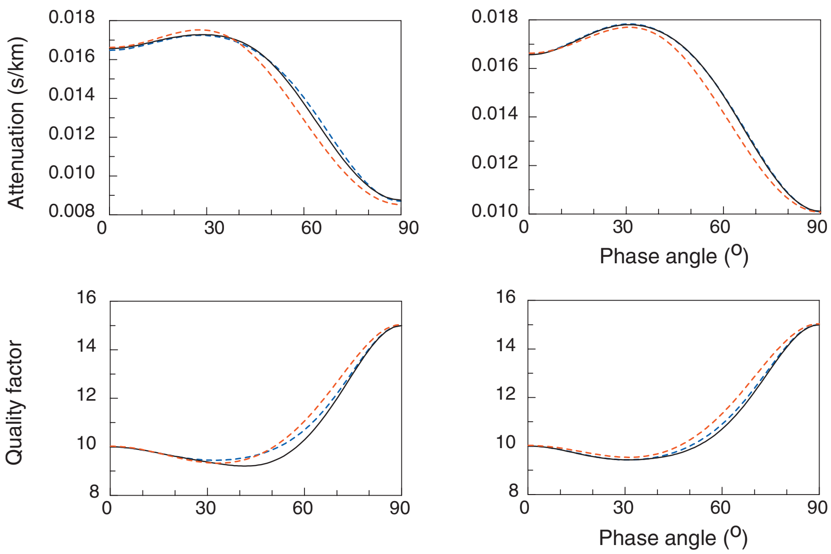

Figure 3. Exact and approximate attenuations and Q-factors in models A2 (left-hand plots) and B2 (right-hand plots). Black solid lines show exact phase quantities. Blue dashed lines show approximate quantities of the improved accuracy calculated using equations 31, 38, and 39. Red dashed lines show approximate quantities calculated using equations 41 and 42. The phase angle denotes the deviation of the real part of the complex slowness vector from the symmetry axis.

DISCUSSION

Numerical modeling shows that perturbation equations differ in accuracy for different wave quantities. The propagation velocity usually is calculated with high accuracy. However, the attenuation and Q-factor might be reproduced with appreciably lower accuracy. This happens mostly when the anisotropy strength is higher than 10% and the attenuation is moderate or weak ($Q > 20$). In this case, first-order perturbations might appear to be an approximation that is too inaccurate, so a modified approach would be required. To overcome this difficulty, it is possible to introduce real weak attenuation parameters in a slightly more complicated form than defined in this paper. This is done by Zhu and Tsvankin (2006, 2007) for TI and orthorhombic anisotropy. These definitions automatically include some effects of the velocity anisotropy (i.e., weak attenuation parameters depend on weak velocity parameters).

Alternatively, we can incorporate some higher-order perturbations into equations for attenuation and the Q-factor by calculating the slowness direction in an actual anisotropic medium but not in an isotropic reference medium (see equation 10). The numerical examples prove that this approach is more accurate than the linearized approach by Zhu and Tsvankin (2006).

Finally, it is possible to use perturbations for evaluating the slowness vector (equations A-8–A-10) and possibly the polarization vector (equations B-7–B-9). All other calculations can be performed exactly. Obviously, this approach yields the most accurate results (Vavryčuk, 2008). Another highly accurate nonlinear approximation for the attenuation coefficient in TI media is given by Zhu and Tsvankin (2006; their equation 19).

CONCLUSIONS

WAA parameters are an effective tool for calculating wave quantities in weakly anisotropic attenuating media of arbitrary symmetry. The WAA parameters can be introduced alternatively as complex-valued or real-valued quantities. The use of complex-valued WAA parameters seems to be mathematically more elegant and less laborious when writing computer code, but real-valued WAA parameters are probably more comprehensible and their physical meaning is more understandable. For example, the velocity anisotropy parameters are very similar to linearized versions of Thomsen parameters used widely in seismic processing and inversion in TI media. Thomsen parameters use a fixed reference medium, whereas velocity anisotropy parameters use a reference medium that can be adjusted. Because first-order perturbation equations of the wave quantities depend linearly on WAA parameters, the WAA parameters can be calculated easily in inverse problems.

The perturbation approach also has its limitations. First, it is limited by the strength of anisotropy and attenuation. Perturbations work well in anisotropic media where the phase and ray quantities are not very different because first-order perturbations do not distinguish between phase and ray quantities. Obviously, perturbations are not applicable to media with strong anisotropy or anisotropy displaying triplications. Standard perturbation equations do not work near singularities (acoustic axes) where the Christoffel tensor becomes nearly degenerate. In this case, perturbation equations must be modified.

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APPENDIX A

PERTURBATION OF THE SLOWNESS VECTOR

The slowness vector is taken at a stationary point on the slowness surface and calculated using first-order perturbations. The stationary point is a point for which energy velocity vector **v** is homogeneous and its direction is parallel to the ray. The approach is basically the same as that presented in Vavryčuk (2008). However, instead of an anisotropic elastic medium assumed in Vavryčuk (2008), an isotropic viscoelastic medium is considered.

The perturbation of the P-wave stationary slowness vector for the anisotropic viscoelastic reference medium reads (see equation 38 in Vavryčuk, 2008)

$$\Delta p_l^{(1)} = [H_{il}^{0(1)}]^{-1}\left(\frac{v_i^{0(1)}}{2}\Delta a_{mjkl} p_m^{0(1)} p_l^{0(1)} g_j^{0(1)} g_k^{0(1)} - \Delta_a v_i^{(1)}\right), \quad \text{(A-1)}$$

where

$$H_{il}^{0(1)} = a_{ijkl}^{0} g_j^{0(1)} g_k^{0(1)} + \frac{v_i^{0(12)} v_l^{0(12)}}{G^{0(1)} - G^{0(2)}} + \frac{v_i^{0(13)} v_l^{0(13)}}{G^{0(1)} - G^{0(3)}}, \quad \text{(A-2)}$$

$$\Delta_a v_i^{(1)} = \Delta a_{mjkl} p_l^{0(1)} g_j^{0(1)} \left[\delta_{im} g_k^{0(1)} + \frac{v_i^{0(12)} p_m^{0(1)} g_k^{0(2)}}{G^{0(1)} - G^{0(2)}} + \frac{v_i^{0(13)} p_m^{0(1)} g_k^{0(3)}}{G^{0(1)} - G^{0(3)}} \right], \quad \text{(A-3)}$$

$$v_i^{0(12)} = a_{ijkl}^{0} p_l^{0(1)} (g_j^{0(1)} g_k^{0(2)} + g_j^{0(2)} g_k^{0(1)}), \quad \text{(A-4)}$$

$$v_i^{0(13)} = a_{ijkl}^{0} p_l^{0(1)} (g_j^{0(1)} g_k^{0(3)} + g_j^{0(3)} g_k^{0(1)}), \quad \text{(A-5)}$$

and δ_{ij} is the Kronecker delta. The superscript (1, 2, and 3) in parentheses means the type of wave (P, S1, and S2). Quantity $H_{il}^{0(1)}$ is the P-wave metric tensor of the reference medium (Vavryčuk, 2003). The equations for the S1- and S2-wave stationary slowness vectors are analogous. Taking into account that in isotropic media

$$H_{il}^{0} = c_0^2 \delta_{il}, \quad [H_{il}^{0}]^{-1} = c_0^{-2} \delta_{il},$$

$$v_i^{0(12)} g_i^{0(1)} = 0, \quad v_i^{0(12)} g_i^{0(2)} = \frac{(c_0^{\mathrm{P}})^2 - (c_0^{\mathrm{S}})^2}{c_0},$$

$$v_i^{0(12)} g_i^{0(3)} = 0,$$

$$v_i^{0(13)} g_i^{0(1)} = 0, \quad v_i^{0(13)} g_i^{0(2)} = 0,$$

$$v_i^{0(13)} g_i^{0(3)} = \frac{(c_0^{\mathrm{P}})^2 - (c_0^{\mathrm{S}})^2}{c_0},$$

$$v_i^{0(23)} g_i^{0(1)} = 0, \quad v_i^{0(23)} g_i^{0(2)} = 0, \quad v_i^{0(23)} g_i^{0(3)} = 0, \quad \text{(A-6)}$$

where c_0 stands for c_0^{P} or c_0^{S}, depending on the wave studied, we obtain

$$\Delta_a v_m^{(1)} g_m^{0(1)} = \frac{1}{c_0^{\mathrm{P}}} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 n_k^0,$$

$$\Delta_a v_m^{(1)} g_m^{0(2)} = \frac{2}{c_0^{\mathrm{P}}} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(2)},$$

$$\Delta_a v_m^{(1)} g_m^{0(3)} = \frac{2}{c_0^{\mathrm{P}}} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(3)},$$

$$\Delta_a v_m^{(2)} g_m^{0(1)} = \frac{1}{c_0^{\mathrm{S}}} \Delta a_{ijkl} n_i^0 n_l^0 g_j^{0(2)} g_k^{0(2)},$$

$$\Delta_a v_m^{(2)} g_m^{0(2)} = \frac{1}{c_0^{\mathrm{S}}} \Delta a_{ijkl} n_l^0 g_j^{0(2)} (g_i^{0(2)} g_k^{0(2)} - n_i^0 n_k^0),$$

$$\Delta_a v_m^{(2)} g_m^{0(3)} = \frac{1}{c_0^{\mathrm{S}}} \Delta a_{ijkl} n_i^0 g_j^{0(2)} g_k^{0(2)} g_l^{0(3)},$$

$$\Delta_a v_m^{(3)} g_m^{0(1)} = \frac{1}{c_0^{\mathrm{S}}} \Delta a_{ijkl} n_i^0 n_l^0 g_j^{0(3)} g_k^{0(3)},$$

$$\Delta_a v_m^{(3)} g_m^{0(3)} = \frac{1}{c_0^{\mathrm{S}}} \Delta a_{ijkl} n_l^0 g_j^{0(3)} (g_i^{0(3)} g_k^{0(3)} - n_i^0 n_k^0),$$

$$\Delta_a v_m^{(3)} g_m^{0(2)} = \frac{1}{c_0^{\mathrm{S}}} \Delta a_{ijkl} n_i^0 g_j^{0(3)} g_k^{0(3)} g_l^{0(2)}, \quad \text{(A-7)}$$

and finally

$$\Delta p_m^{(1)} = -\frac{1}{2(c_0^{\mathrm{P}})^3} \Delta a_{ijkl} n_l^0 n_j^0 n_k^0 (4\delta_{im} - 3 n_i^0 n_m^0), \quad \text{(A-8)}$$

$$\Delta p_m^{(2)} = -\frac{1}{2(c_0^{\mathrm{S}})^3} \Delta a_{ijkl} n_l^0 g_j^{0(2)} [g_k^{0(2)} (2\delta_{im} - n_i^0 n_m^0) - 2 n_i^0 n_k^0 g_m^{0(2)}], \quad \text{(A-9)}$$

$$\Delta p_m^{(3)} = -\frac{1}{2(c_0^{\mathrm{S}})^3} \Delta a_{ijkl} n_l^0 g_j^{0(3)} [g_k^{0(3)} (2\delta_{im} - n_i^0 n_m^0) - 2 n_i^0 n_k^0 g_m^{0(3)}]. \quad \text{(A-10)}$$

It follows from equations A-8–A-10 that if perturbations Δa_{ijkl} are real valued, the perturbations of the slowness vectors $\Delta \mathbf{p}^{(1)}$, $\Delta \mathbf{p}^{(2)}$, and $\Delta \mathbf{p}^{(3)}$ are likewise real valued. This means a weakly anisotropic medium with isotropic attenuation and a weakly anisotropic elastic medium generates a homogeneous stationary slowness vector.

For the perturbation of the slowness direction, we readily obtain

$$\Delta n_m^{(1)} = -\frac{2}{(c_0^{\mathrm{P}})^2} \Delta a_{ijkl} n_l^0 n_j^0 n_k^0 (\delta_{im} - n_i^0 n_m^0), \quad \text{(A-11)}$$

$$\Delta n_m^{(2)} = -\frac{1}{(c_0^{\mathrm{S}})^2} \Delta a_{ijkl} n_l^0 g_j^{0(2)} [g_k^{0(2)} (\delta_{im} - n_i^0 n_m^0) - n_i^0 n_k^0 g_m^{0(2)}], \quad \text{(A-12)}$$

$$\Delta n_m^{(3)} = -\frac{1}{(c_0^{\mathrm{S}})^2} \Delta a_{ijkl} n_l^0 g_j^{0(3)} [g_k^{0(3)} (\delta_{im} - n_i^0 n_m^0) - n_i^0 n_k^0 g_m^{0(3)}]. \quad \text{(A-13)}$$

APPENDIX B

PERTURBATION OF THE POLARIZATION VECTOR

Perturbation of the P-wave eigenvector $\mathbf{g}^{(1)}$ of the Christoffel tensor Γ_{jk} is expressed as a sum of perturbations projected into the directions of the S-wave polarization vectors $\mathbf{g}^{0(2)}$ and $\mathbf{g}^{0(3)}$:

$$\Delta g_i^{(1)} = \frac{\Delta G^{(12)}}{G^{0(1)} - G^{0(2)}} g_i^{0(2)} + \frac{\Delta G^{(13)}}{G^{0(1)} - G^{0(3)}} g_i^{0(3)}, \quad \text{(B-1)}$$

where

$$\Delta G^{(rs)} = \Delta \Gamma_{jk} g_j^{0(r)} g_k^{0(s)}, \quad \text{(B-2)}$$

$$\Delta \Gamma_{jk} = \Delta a_{ijkl} p_i^0 p_l^0 + a_{ijkl}^0 (p_i^0 \Delta p_l + p_l^0 \Delta p_i). \qquad \text{(B-3)}$$

Taking into account equation A-8, we can write

$$\Delta p_m^{(1)} g_m^{0(1)} = -\frac{1}{2(c_0^{\mathrm{P}})^3} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 n_k^0,$$

$$\Delta p_m^{(1)} g_m^{0(2)} = -\frac{2}{(c_0^{\mathrm{P}})^3} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(2)},$$

$$\Delta p_m^{(1)} g_m^{0(3)} = -\frac{2}{(c_0^{\mathrm{P}})^3} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(3)}. \qquad \text{(B-4)}$$

Consequently, from equations B-2 and B-3, we obtain

$$\Delta \Gamma_{jk} g_j^{0(1)} g_k^{0(2)} = \frac{2(c_0^{\mathrm{S}})^2 - (c_0^{\mathrm{P}})^2}{(c_0^{\mathrm{P}})^4} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(2)}, \qquad \text{(B-5)}$$

$$\Delta \Gamma_{jk} g_j^{0(1)} g_k^{0(3)} = \frac{2(c_0^{\mathrm{S}})^2 - (c_0^{\mathrm{P}})^2}{(c_0^{\mathrm{P}})^4} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(3)}, \qquad \text{(B-6)}$$

and finally

$$\Delta g_m^{(1)} = \frac{1}{(c_0^{\mathrm{P}})^2} \frac{2(c_0^{\mathrm{S}})^2 - (c_0^{\mathrm{P}})^2}{(c_0^{\mathrm{P}})^2 - (c_0^{\mathrm{S}})^2} \Delta a_{ijkl} n_l^0 n_j^0 n_k^0 (\delta_{im} - n_i^0 n_m^0). \qquad \text{(B-7)}$$

The perturbation of the S-wave polarization vectors $\mathbf{g}^{(2)}$ and $\mathbf{g}^{(3)}$ projected into the direction of the P-wave polarization vectors $\mathbf{g}^{0(1)}$ can be found in an analogous way. We obtain

$$\Delta g_m^{(2)} = \left\{ \frac{1}{(c_0^{\mathrm{S}})^2} \Delta a_{ijkl} n_l^0 g_j^{0(2)} (g_i^{0(2)} g_k^{0(2)} - n_i^0 n_k^0) - \frac{1}{(c_0^{\mathrm{P}})^2 - (c_0^{\mathrm{S}})^2} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(2)} \right\} n_m^0 \qquad \text{(B-8)}$$

and

$$\Delta g_m^{(3)} = \left\{ \frac{1}{(c_0^{\mathrm{S}})^2} \Delta a_{ijkl} n_l^0 g_j^{0(3)} (g_i^{0(3)} g_k^{0(3)} - n_i^0 n_k^0) - \frac{1}{(c_0^{\mathrm{P}})^2 - (c_0^{\mathrm{S}})^2} \Delta a_{ijkl} n_i^0 n_l^0 n_j^0 g_k^{0(3)} \right\} n_m^0. \qquad \text{(B-9)}$$

Because the isotropic reference medium is degenerate for the S-waves, the perturbation of polarization vectors $\mathbf{g}^{(2)}$ and $\mathbf{g}^{(3)}$ projected into the $\mathbf{g}^{0(2)} - \mathbf{g}^{0(3)}$ plane is calculated in a more complicated way (see Farra, 2001; Appendix A of Vavryčuk, 2003) and is not presented here. For a TI medium, the projections of the SH- and SV-waves are identically zero.

APPENDIX C

PERTURBATION OF THE POLARIZATION VECTOR, SLOWNESS VECTOR, AND SLOWNESS DIRECTION IN TI MEDIA

Perturbation equations for stationary slowness vector $\Delta \mathbf{p}$ (Appendix A), its direction $\Delta \mathbf{n}$ (Appendix A), and polarization vector $\Delta \mathbf{g}$ (Appendix B) are simplified in TI media. Substituting Δa_{ijkl} for TI and taking into account that the S1- and S2-waves become the SH- and SV-waves in TI media, we obtain for the P-wave

$$\Delta p_1^{\mathrm{P}} = N_1 C_p^{\mathrm{P}} [3A_1 N_3^4 + 2A_2 N_3^2 + \Delta a_{11}],$$
$$\Delta p_3^{\mathrm{P}} = N_3 C_p^{\mathrm{P}} [3A_1 N_3^4 + 2(A_2 - 2A_1) N_3^2 - 4A_2 + \Delta a_{11}], \qquad \text{(C-1)}$$

$$\Delta n_1^{\mathrm{P}} = N_1 C_n^{\mathrm{P}} [A_1 N_3^4 + A_2 N_3^2],$$
$$\Delta n_3^{\mathrm{P}} = N_3 C_n^{\mathrm{P}} [A_1 N_3^4 + (A_2 - A_1) N_3^2 - A_2], \qquad \text{(C-2)}$$

$$\Delta g_1^{\mathrm{P}} = N_1 C_g^{\mathrm{P}} [A_1 N_3^4 + A_2 N_3^2],$$
$$\Delta g_3^{\mathrm{P}} = N_3 C_g^{\mathrm{P}} [A_1 N_3^4 + (A_2 - A_1) N_3^2 - A_2], \qquad \text{(C-3)}$$

where

$$C_p^{\mathrm{P}} = -\frac{1}{2(c_0^{\mathrm{P}})^3}, \quad C_n^{\mathrm{P}} = -\frac{2}{(c_0^{\mathrm{P}})^2},$$
$$C_g^{\mathrm{P}} = \frac{1}{(c_0^{\mathrm{P}})^2} \frac{2(c_0^{\mathrm{S}})^2 - (c_0^{\mathrm{P}})^2}{(c_0^{\mathrm{P}})^2 - (c_0^{\mathrm{S}})^2}. \qquad \text{(C-4)}$$

For the SV-wave,

$$\Delta p_1^{\mathrm{SV}} = N_1 C_p^{\mathrm{SV}} [A_1 N_3^2 (1 - 3N_3^2) + \Delta a_{44}],$$
$$\Delta p_3^{\mathrm{SV}} = N_3 C_p^{\mathrm{SV}} [A_1 N_3^2 (5 - 3N_3^2) - 2A_1 + \Delta a_{44}], \qquad \text{(C-5)}$$

$$\Delta n_1^{\mathrm{SV}} = N_1 C_n^{\mathrm{SV}} A_1 N_3^2 (N_1^2 - N_3^2),$$
$$\Delta n_3^{\mathrm{SV}} = -N_3 C_n^{\mathrm{SV}} A_1 N_1^2 (N_1^2 - N_3^2), \qquad \text{(C-6)}$$

$$\Delta g_1^{\mathrm{SV}} = \Delta n_3^{\mathrm{SV}} + N_3 C_g^{\mathrm{SV}} \times [A_1 N_3^4 + (A_2 - A_1) N_3^2 - A_2],$$
$$\Delta g_3^{\mathrm{SV}} = -\Delta n_1^{\mathrm{SV}} - N_1 C_g^{\mathrm{SV}} [A_1 N_3^4 + A_2 N_3^2], \qquad \text{(C-7)}$$

where

$$C_p^{\mathrm{SV}} = -\frac{1}{2(c_0^{S})^3}, \quad C_n^{\mathrm{SV}} = -\frac{1}{(c_0^{S})^2}, \quad C_g^{\mathrm{SV}} = \frac{1}{(c_0^{P})^2 - (c_0^{S})^2}. \qquad \text{(C-8)}$$

Using WAA parameters, the equations for constants A_1 and A_2 are expressed in terms of perturbations Δa_{ijkl} as

$$A_1 = -\Delta a_{11} + 2\Delta a_{13} - \Delta a_{33} + 4\Delta a_{44},$$
$$A_2 = \Delta a_{11} - \Delta a_{13} - 2\Delta a_{44}. \qquad \text{(C-9)}$$

In terms of WAA parameters, they are expressed as

$$A_1 = 2(c_0^{\mathrm{P}})^2(\delta_x - \varepsilon_x - \varepsilon_z), \quad A_2 = (c_0^{\mathrm{P}})^2(-\delta_x + 2\varepsilon_x). \tag{C-10}$$

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