GEOPHYSICS, VOL. 80, NO. 3 (MAY-JUNE 2015); P. C59–C71, 6 FIGS., 3 TABLES.
10.1190/GEO2014-0355.1

Determination of parameters of viscoelastic anisotropy from ray velocity and ray attenuation: Theory and numerical modeling

Václav Vavryčuk[1]

ABSTRACT

We have developed and numerically tested a method for determining parameters of homogeneous viscoelastic anisotropy from measurements of wavefields generated by point sources. The method is based on complex algebra and consists of several steps. First, a complex energy velocity surface is constructed from the directionally dependent velocity and attenuation measured along a set of ray directions. Second, a complex slowness surface is computed using the relation of polar reciprocity between the energy velocity and slowness vectors. The energy velocity vectors are homogeneous, but the corresponding slowness vectors are inhomogeneous. Finally, the complex phase velocity surface is calculated and inverted using the Christoffel equation. The inversion is nonlinear and can be performed in iterations. Numerical tests for the P-wave in transversely isotropic media showed that the method performed well for a wide range of models covering strong as well as weak velocity anisotropy and various levels of attenuation. The method was compared with a simplified approximate inversion when the inhomogeneity of the complex slowness vector is neglected. The neglect of the slowness vector inhomogeneity results in a significantly lower accuracy of the retrieved attenuation parameters. Accuracy with errors less than 10% is achieved only if the attenuation anisotropy is weak. This condition is, however, strongly restrictive because attenuation anisotropy is usually significant being more pronounced than the velocity anisotropy for most of rocks.

INTRODUCTION

Seismic observations confirm that most rocks are anisotropic and attenuating (Burton, 2007; Carcione, 2007). Seismic anisotropy can be intrinsic if produced by anisotropic mineral grains (Babuška and Cara, 1991; Karato, 2008), but also effective if produced by the presence of layers (Backus, 1962; Picotti et al., 2010), preferentially oriented small-scale inhomogeneities or joints, cracks, and microcracks (Hudson, 1981; Schoenberg and Douma, 1988). The presence of structure complexities in rocks affects not only velocities of seismic waves but also their amplitudes. The seismic waves are scattered, and their energy is dissipated. Hence, anisotropy and attenuation are inherently bound and jointly affect the seismic waves.

In anisotropic attenuating media, we distinguish between velocity and attenuation anisotropy. Velocity anisotropy characterizes a directionally dependent propagation velocity of seismic waves, whereas attenuation anisotropy controls directionally dependent dissipation of seismic energy and, consequently, a decay of wave amplitudes along a ray. Both wave phenomena are studied using a model of an anisotropic viscoelastic medium described by complex-valued, frequency-dependent, viscoelastic parameters (Auld, 1973; Carcione, 2007). The real part of these parameters controls the propagation of waves and their imaginary part controls the wave attenuation. The equations for the propagation of wavefields in viscoelastic anisotropic media are formally the same as in elastic media except for being complex. Involving the complex algebra into equations is not an essential complication, but still it means taking some care for understanding properly the physical meaning of all complex-valued quantities standing in the equations and for elaborating correctly with them.

In studies of wave propagation in viscoelastic anisotropy, the focus has been mostly put on plane waves (Carcione and Cavallini, 1993; Deschamps et al., 1997; Shuvalov and Scott, 1999; Deschamps and Assouline, 2000, Červený and Pšenčík, 2005; Zhu and Tsvankin, 2006, 2007; Picotti et al., 2010; Rasolofosaon,

Manuscript received by the Editor 1 August 2014; revised manuscript received 6 December 2014; published online 1 April 2015.
[1]Academy of Sciences of the Czech Republic, Institute of Geophysics, Praha, Czech Republic. E-mail: vv@ig.cas.cz.

2010). Less attention has been paid to the more complicated topic of waves generated by point sources. The first works were devoted to theoretical analysis of wavefronts (Carcione, 1994), modeling of high-frequency wavefields by the ray method (Hearn and Krebes, 1990; Gajewski and Pšenčík, 1992; Zhu and Chun, 1994), and modeling of complete wavefields by directly solving the wave equation (Carcione, 1990, 1993; Carcione et al., 1996). Later, exact and asymptotic Green's functions for a homogeneous medium are derived (Vavryčuk, 2007a), and exact and approximate solutions of the complex eikonal equation special type of an inhomogeneous medium are reported (Vavryčuk, 2012). It is first recognized by Vavryčuk (2007a, 2007b) that the phase and group quantities describing attenuation anisotropy should be distinguished, similar to how we distinguish between the phase and group velocities in elastic anisotropy.

In this paper, we study properties of wavefields generated by point sources and propagating in anisotropic viscoelastic media. We focus on developing a method for determining parameters of viscoelastic anisotropy from measurements of the velocity and attenuation of waves along a ray. The method is suitable for the far-field measurements in situ or in labs when the sensors are at a distance larger than 10 wavelengths from the source. This condition is equivalent to the so-called high-frequency approximation commonly used in ray theory. An experimental setup suitable for such measurements is described, for example, in Svitek et al. (2014). Because attenuating media are inherently dispersive, we do not use the terms group velocity and group attenuation for quantities measured along a ray. The "group" quantities in dispersive media describe the propagation of wave packets with a broad spectrum of frequencies but not the propagation of monochromatic waves. Therefore, the terms ray velocity and ray attenuation are consistently used in this paper. We show how to solve the inverse problem using complex algebra. On numerical examples of P-waves propagating in transversely isotropic media, we demonstrate the effectiveness and accuracy of the developed methods.

WAVE PROPAGATION IN ANISOTROPIC VISCOELASTIC MEDIA

In this section, basic formulas for waves propagating in anisotropic viscoelastic media are reviewed. In the formulas, real and imaginary parts of complex-valued quantities are denoted by superscripts R and I, respectively. A complex-conjugate quantity is denoted by an asterisk. The direction of a complex-valued vector $\mathbf{v}$ is calculated as $\mathbf{v}/\sqrt{\mathbf{v}\cdot\mathbf{v}}$, where the dot means the scalar product (the normalization condition $\mathbf{v}/\sqrt{\mathbf{v}\cdot\mathbf{v}^*}$ is not used because it complicates generalizing some of real equations to complex ones). The magnitude of complex-valued vector $\mathbf{v}$ is $\sqrt{\mathbf{v}\cdot\mathbf{v}}$. If any complex-valued vector is defined by a real-valued direction, it is called *homogeneous*, and if it is defined by a complex-valued direction, it is called *inhomogeneous*. The terms homogeneous and inhomogeneous are also used for characterizing spatially independent or spatially dependent viscoelastic parameters of a medium and for characterizing wavefronts of plane waves with a constant or exponentially decaying amplitude. The specific meaning of the terms homogeneous and inhomogeneous is clear from the context. In formulas, the Einstein summation convention is used for repeated subscripts.

Basic formulas

A viscoelastic anisotropic medium is defined by density-normalized stiffness parameters $a_{ijkl} = c_{ijkl}/\rho$, which are, in general, frequency dependent and complex valued. The real and imaginary parts of a_{ijkl},

$$a_{ijkl}(\omega) = a^R_{ijkl} + i a^I_{ijkl}, \tag{1}$$

define the elastic and viscous properties of the medium. Consequently, the Christoffel tensor Γ_{jk}, defined alternatively in terms of slowness direction $\mathbf{n}$,

$$\Gamma_{jk}(\mathbf{n}) = a_{ijkl} n_i n_l, \tag{2}$$

or slowness vector $\mathbf{p}$,

$$\Gamma_{jk}(\mathbf{p}) = a_{ijkl} p_i p_l, \tag{3}$$

is frequency dependent and complex. Slowness direction $\mathbf{n}$ is real for homogeneous plane waves but complex for inhomogeneous plane waves. The eigenvalues of the Christoffel tensor G are calculated using the following equation:

$$\det(\Gamma_{jk} - G\delta_{jk}) = 0, \tag{4}$$

which yields the cubic equation in G

$$G^3 - PG^2 + QG - R = 0, \tag{5}$$

where P, Q, and R are defined as

$$\begin{aligned} P &= \Gamma_{11} + \Gamma_{22} + \Gamma_{33}, \\ Q &= \Gamma_{11}\Gamma_{22} + \Gamma_{11}\Gamma_{33} + \Gamma_{22}\Gamma_{33} - \Gamma_{12}^2 - \Gamma_{13}^2 - \Gamma_{23}^2, \\ R &= \Gamma_{11}\Gamma_{22}\Gamma_{33} + 2\Gamma_{12}\Gamma_{13}\Gamma_{23} - \Gamma_{11}\Gamma_{23}^2 - \Gamma_{22}\Gamma_{13}^2 - \Gamma_{33}\Gamma_{12}^2. \end{aligned} \tag{6}$$

Alternatively, the eigenvalues can be expressed as

$$G(\mathbf{n}) = a_{ijkl} n_i n_l g_j g_k = c^2 \tag{7}$$

and

$$G(\mathbf{p}) = a_{ijkl} p_i p_l g_j g_k = 1. \tag{8}$$

Equations 7 and 8 define phase velocity c and slowness vector $\mathbf{p} = \mathbf{n}/c$ as a function of slowness direction $\mathbf{n}$. The eigenvectors define polarization vectors $\mathbf{g}$. The polarization vectors are normalized, so that $\mathbf{g}\cdot\mathbf{g} = 1$. We emphasize that a more common normalization in complex algebra $\mathbf{g}\cdot\mathbf{g}^* = 1$ is not used because it leads to inconsistencies between equations in elastic and viscoelastic media.

Differentiating eigenvalue $G(\mathbf{p})$, we define the energy velocity vector as

$$v_i = \frac{1}{2}\frac{\partial G}{\partial p_i} = a_{ijkl} p_l g_j g_k, \tag{9}$$

which is called the group velocity vector in elastic media. Vectors $\mathbf{v}$ and $\mathbf{p}$ are related by the equation

$$\mathbf{v} \cdot \mathbf{p} = 1, \tag{10}$$

expressing their polar reciprocity (Helbig, 1994). The slowness vector, phase velocity, energy velocity, and the polarization vectors are, in general, complex.

Phase velocity, attenuation, and Q-factor

Phase quantities describe the propagation of plane waves. Decomposing the complex slowness vector $\mathbf{p}$ as follows (see Vavryčuk [2007b], equation 2):

$$\mathbf{p} = \left[\frac{1}{V^{\text{phase}}} + iA^{\text{phase}}\right]\mathbf{s} + iD^{\text{phase}}\,\mathbf{t}, \tag{11}$$

and inserting it into the exponential term $\exp(i\omega\,\mathbf{p}\cdot\mathbf{x})$ describing the propagation of harmonic waves, we obtain

$$\exp(i\omega\mathbf{p}\cdot\mathbf{x}) = \exp(-\omega A^{\text{phase}}\mathbf{s}\cdot\mathbf{x})\exp(-\omega D^{\text{phase}}\mathbf{t}\cdot\mathbf{x}) \times \exp\left(i\omega\frac{\mathbf{s}\cdot\mathbf{x}}{V^{\text{phase}}}\right), \tag{12}$$

where ω is the circular frequency; $\mathbf{x}$ is the position vector of the receiver; and quantities V^{phase}, A^{phase}, and D^{phase} are the real phase velocity, phase attenuation, and phase inhomogeneity. Vectors $\mathbf{s}$ and $\mathbf{t}$ are real, mutually perpendicular unit vectors; $\mathbf{s}$ is normal to the wavefront (called the *wave normal*); and $\mathbf{t}$ lies in the wavefront (called the *wave tangent*). Hence, the phase velocity, attenuation, and inhomogeneity are calculated from $\mathbf{p}$ as

$$V^{\text{phase}} = \frac{1}{|\mathbf{p}^R|}, \quad A^{\text{phase}} = \mathbf{p}^I\cdot\mathbf{s}, \quad D^{\text{phase}} = \mathbf{p}^I\cdot\mathbf{t}, \tag{13}$$

where

$$\mathbf{s} = \frac{\mathbf{p}^R}{|\mathbf{p}^R|}, \qquad \mathbf{t} = \frac{\mathbf{p}^I - (\mathbf{p}^I\cdot\mathbf{s})\mathbf{s}}{|\mathbf{p}^I - (\mathbf{p}^I\cdot\mathbf{s})\mathbf{s}|}, \tag{14}$$

and symbol $|\mathbf{a}| = \sqrt{\mathbf{a}\cdot\mathbf{a}} = \sqrt{a_j a_j}$ denotes the magnitude of real vector $\mathbf{a}$.

For a homogeneous plane wave ($D^{\text{phase}} = 0$), the phase velocity and attenuation can be calculated from the complex phase velocity using the following formulas:

$$V^{\text{phase}} = \frac{c^R c^R + c^I c^I}{c^R}, \qquad A^{\text{phase}} = -\frac{c^I}{c^R c^R + c^I c^I}. \tag{15}$$

The phase velocity V^{phase} and phase attenuation A^{phase} are intrinsic properties of the medium for homogeneous waves because they depend on viscoelastic parameters of the medium and on the wave normal only. The phase inhomogeneity D^{phase} is a free parameter that depends on the specification of boundary conditions of each wave propagation problem.

Similar to the phase attenuation A^{phase}, the phase quality factor (hereinafter the Q^{phase}-factor) is also an intrinsic property of the medium. It is defined for homogeneous plane waves only and takes the following form (Carcione [2000], his equation 14; Carcione [2007], his equation 4.92; Chichinina et al. [2006], their equation 27):

$$Q^{\text{phase}} = -\frac{(c^2)^R}{(c^2)^I}, \tag{16}$$

where c is the complex phase velocity, $c = 1/\sqrt{p_i p_i}$.

Ray velocity, attenuation, and Q-factor

Decomposing the exponential term describing the propagation of high-frequency harmonic waves from a point source, we obtain

$$\exp(i\omega\mathbf{p}\cdot\mathbf{x}) = \exp(i\omega\frac{r}{v}) = \exp(-\omega A^{\text{ray}} r)\exp\left(i\omega\frac{r}{V^{\text{ray}}}\right), \tag{17}$$

where ω is the circular frequency, r is the distance between the source and the receiver, and v is the complex energy velocity. Ray velocity V^{ray} and ray attenuation A^{ray} in equation 17 are expressed as (see Vavryčuk [2007b], his equations 21 and 22)

$$V^{\text{ray}} = \frac{v^R v^R + v^I v^I}{v^R}, \qquad A^{\text{ray}} = -\frac{v^I}{v^R v^R + v^I v^I}. \tag{18}$$

The ray velocity controls the propagation velocity along a ray, and the ray attenuation controls the amplitude decay along a ray. The ray velocity and ray attenuation are real and can be observed and measured in wavefields along a ray. In analogy to the phase quality factor defined in equation 16 using the complex phase velocity, we can also introduce the ray quality factor using the complex energy velocity (see Vavryčuk [2007b], his equation 24):

$$Q^{\text{ray}} = -\frac{(v^2)^R}{(v^2)^I}. \tag{19}$$

Introducing ray inhomogeneity D^{ray} is unnecessary because the complex energy velocity vector $\mathbf{v}$ is homogeneous and D^{ray} is identically zero in homogeneous anisotropic viscoelastic media. In inhomogeneous media, the ray inhomogeneity D^{ray} can be nonzero.

Calculating the complex energy velocity vector $\mathbf{v}$ needed in equations 17–19 is not straightforward. It requires computing the so-called stationary slowness vector $\mathbf{p}_0$ introduced in ray theory (Vavryčuk, 2007a). The stationary slowness vector is generally inhomogeneous being uniquely constrained by the point-source boundary conditions. To satisfy these conditions, we have to solve for each ray coming out of the source either a system of polynomial equations for unknown components of $\mathbf{p}_0$ or an inverse problem for $\mathbf{p}_0$ using iterations (for details, see Vavryčuk, 2007a, 2007b; Grechka, 2015).

Relation between the attenuation and the Q-factor

The relations among the complex velocity, real velocity, and attenuation are for the intrinsic phase quantities and for the ray quantities symmetrical:

$$\frac{1}{c} = \frac{1}{V^{\text{phase}}} + iA^{\text{phase}}, \qquad \frac{1}{v} = \frac{1}{V^{\text{ray}}} + iA^{\text{ray}}. \tag{20}$$

Taking into account equations 16 and 19, we also obtain fully symmetrical relations between the complex velocity and Q-factor:

$$c^2 = (c^2)^R \left(1 - \frac{i}{Q^{\text{phase}}}\right), \qquad v^2 = (v^2)^R \left(1 - \frac{i}{Q^{\text{ray}}}\right). \tag{21}$$

Equations 20 and 21 yield

$$Q^{\text{phase}} = \frac{1 - (A^{\text{phase}} V^{\text{phase}})^2}{2A^{\text{phase}} V^{\text{phase}}}, \qquad Q^{\text{ray}} = \frac{1 - (A^{\text{ray}} V^{\text{ray}})^2}{2A^{\text{ray}} V^{\text{ray}}}. \tag{22}$$

Assuming weak attenuation, $A^{\text{ray}} V^{\text{ray}} \ll 1$, or equivalently, $A^{\text{phase}} V^{\text{phase}} \ll 1$, and neglecting the terms of the second order, we get the equations consistent with equations 51 and 59 of Vavryčuk (2008):

$$Q^{\text{phase}} = \frac{1}{2A^{\text{phase}} V^{\text{phase}}}, \qquad Q^{\text{ray}} = \frac{1}{2A^{\text{ray}} V^{\text{ray}}}, \tag{23}$$

and the equations

$$c = V^{\text{phase}} \sqrt{1 - \frac{i}{Q^{\text{phase}}}}, \qquad v = V^{\text{ray}} \sqrt{1 - \frac{i}{Q^{\text{ray}}}}, \tag{24}$$

useful for constructing the complex velocity from the real velocity and Q-factor. Note that the condition of weak attenuation is different from the condition of weak attenuation anisotropy. Weak attenuation anisotropy requires that the directional dependence of attenuation is weak. Hence, rocks can be weakly attenuating but displaying strong attenuation anisotropy.

DETERMINATION OF PARAMETERS OF GENERAL VISCOELASTIC ANISOTROPY

The goal of this section is to present a procedure for determining parameters of homogeneous anisotropic attenuating media from a directionally dependent propagation velocity and attenuation measured along a ray.

Calculation of complex phase velocity

First, we calculate the complex energy velocity v from the measured real ray velocity V^{ray} and real ray attenuation A^{ray} using equation 20. Subsequently, we obtain vector $\mathbf{v} = v\mathbf{N}$, where $\mathbf{N}$ is the real ray direction, and construct the complex energy velocity surface $v = v(\mathbf{N})$.

Second, we calculate slowness direction $\mathbf{n}$ from the energy velocity surface $v = v(\mathbf{N})$. Because the slowness vector $\mathbf{p}$ and energy velocity vector $\mathbf{v}$ are polar reciprocal (Helbig, 1994), $\mathbf{v}$ is normal to the slowness surface and $\mathbf{p}$ is normal to the energy velocity surface. Hence, slowness direction $\mathbf{n}$ can be calculated as normal to the energy velocity surface using standard formulas of differential geometry (Lipschutz, 1969). Consequently, phase velocity surface $c = c(\mathbf{n})$ is calculated from energy velocity v, ray direction $\mathbf{N}$, and slowness direction $\mathbf{n}$ as follows:

$$c = vN_i n_i, \tag{25}$$

where $\mathbf{N}$ is real but c, v, and $\mathbf{n}$ are complex. Because the slowness direction $\mathbf{n}$ is complex, the slowness vector $\mathbf{p}$ is inhomogeneous.

Inversion scheme

Once the complex phase velocity surface $c = c(\mathbf{n})$ is determined, equation 5 defines the inverse problem for calculating complex viscoelastic parameters a_{ijkl}. Because equation 5 is the cubic equation for 21 unknown parameters, the inversion is nonlinear. It can be solved quite analogously to the inversion for real elastic parameters. The most common approach is to linearize the problem using perturbation theory (Thomsen, 1986; Mensch and Rasolofosaon, 1997; Vavryčuk, 1997; Pšenčík and Gajewski, 1998; Farra and Pšenčík, 2008) and to find the solution in iterations.

In perturbation theory, we assume that the anisotropic medium defined by unknown parameters a_{ijkl} can be obtained by a small perturbation of a known reference medium:

$$a_{ijkl} = a^0_{ijkl} + \Delta a_{ijkl}, \tag{26}$$

where a^0_{ijkl} defines the viscoelastic reference medium and Δa_{ijkl} is its viscoelastic perturbation. Under this assumption, cubic equation 5 for phase velocity c can be linearized as follows (Pšenčík and Vavryčuk, 2002; Svitek et al., 2014):

$$\Delta c^2 = c^2 - c_0^2 = \Delta a_{ijkl} n_i n_l g^0_j g^0_k, \tag{27}$$

where c_0 and $\mathbf{g}^0$ define the complex phase velocity and complex polarization vector in the reference medium and Δc^2 is the misfit between the squares of the phase velocity calculated from measurements and in the reference medium. The reference medium can be isotropic or anisotropic and elastic or viscoelastic.

Equation 27 represents a system of linear equations for unknown complex perturbations Δa_{ijkl}, which can be solved in iterations. If the P-wave velocity and attenuation are inverted and the reference medium is isotropic and viscoelastic in the first iteration, its P-wave velocity α_0 and quality factor Q_0 can be obtained by averaging observed velocities and quality factors over all directions. The complex phase velocity is then calculated as

$$c_0 = \alpha_0 \sqrt{1 - \frac{i}{Q_0}}. \tag{28}$$

Similarly, the complex S-wave velocity in the isotropic reference medium can be supplied. If the mean S-wave velocity and Q-factor are not known, we can use the standard model with ratios between the P- and S-wave velocities and Q-factors, $V_S = V_P/\sqrt{3}$ and $Q_S = Q_P/2$. In higher iterations, the reference medium is the result of the previous iteration and the estimates of the S-wave velocity and attenuation are no longer needed.

Using the above approach, we can invert for all 21 viscoelastic parameters. However, if we invert P-wave quantities only, six parameters related to the S-waves (hereafter the S-wave-related parameters) a_{44}, a_{55}, a_{66}, a_{45}, a_{46}, and a_{56} will not be well resolved. Under weak velocity and attenuation anisotropy, these six parameters cannot be determined from the P-waves at all. Hence, to determine the complete elastic tensor accurately, measurements of the S-wave velocity and attenuation must be included in the inversion.

DETERMINATION OF VISCOELASTIC TRANSVERSE ISOTROPY FROM P-WAVE VELOCITY AND ATTENUATION

Christoffel equation

In transverse isotropy, the procedure is simplified because all directionally dependent quantities are axially symmetric. For simplicity, we use a local coordinate system in which the symmetry axis is along the vertical axis. The medium is described by the following density-normalized stiffness parameters in the Voigt notation: a_{11}, $a_{22}=a_{11}$, a_{33}, a_{44}, $a_{55}=a_{44}$, a_{66}, a_{13}, $a_{23}=a_{13}$, and $a_{12}=a_{11}-2a_{66}$. All other parameters are zero. The parameters a_{ij} are complex. Because the wave quantities are axially symmetric, it is sufficient to study them in the x_1-x_3 plane.

Cubic equation 5 for G splits in transverse isotropy into the quadratic equation for G describing the P- and SV-waves,

$$G^2-(\Gamma_{11}+\Gamma_{33})G+\Gamma_{11}\Gamma_{33}-\Gamma_{13}^2=0, \tag{29}$$

and the linear equation for G describing the SH-wave,

$$G-\Gamma_{22}=0. \tag{30}$$

Inserting equations 2 and 7 into equations 29 and 30, we obtain for the P- or SV-waves

$$a_{11}a_{44}n_1^4+a_{33}a_{44}n_3^4+(a_{11}a_{33}-a_{13}^2-2a_{13}a_{44})n_1^2n_3^2 \\ -(a_{11}+a_{44})n_1^2c^2-(a_{33}+a_{44})n_3^2c^2+c^4=0, \tag{31}$$

and for the SH-wave

$$a_{66}n_1^2+a_{44}n_3^2-c^2=0, \tag{32}$$

or equivalently, for the P- or SV-waves,

$$a_{11}a_{44}\sin^4\vartheta+a_{33}a_{44}\cos^4\vartheta \\ +(a_{11}a_{33}-a_{13}^2-2a_{13}a_{44})\sin^2\vartheta\cos^2\vartheta-(a_{11}+a_{44})c^2\sin^2\vartheta \\ -(a_{33}+a_{44})c^2\cos^2\vartheta+c^4=0, \tag{33}$$

and for the SH-wave

$$a_{66}\sin^2\vartheta+a_{44}\cos^2\vartheta-c^2=0, \tag{34}$$

where slowness direction $\mathbf{n}$ is expressed by slowness angle ϑ, $\mathbf{n}=(\sin\vartheta,0,\cos\vartheta)^T$.

Calculation of complex phase velocity

For determining parameters a_{11}, a_{33}, a_{44}, and a_{13} from measurements of the P-wave using equation 33, we need complex phase velocity $c=c(\vartheta)$ evaluated for a set of slowness angles ϑ. Similarly as for general anisotropy, we calculate complex energy velocity $v=v(\theta)$ from the measured ray velocity and ray attenuation using equation 20. Then, we calculate the complex slowness direction $\mathbf{n}$ as the normal to the energy velocity surface $v=v(\theta)$. Hence, $\mathbf{n}$ is perpendicular to the tangent of surface $v=v(\theta)$:

$$\mathbf{n}\perp\frac{d\mathbf{v}}{d\theta}. \tag{35}$$

Consequently, the slowness angle ϑ is expressed as follows:

$$\vartheta=a\,\cos\left(\frac{dv_1}{d\theta}\left[\left(\frac{dv_1}{d\theta}\right)^2+\left(\frac{dv_3}{d\theta}\right)^2\right]^{-1}\right). \tag{36}$$

The slowness angle is generally complex except for real values of 0° and 90°. In these directions, vectors $\mathbf{n}$ and $\mathbf{N}$ must coincide for symmetry reasons. Because ray direction $\mathbf{N}$ is always real in homogeneous media (Vavryčuk, 2007a), the slowness direction $\mathbf{n}$ must be real if it coincides with $\mathbf{N}$. Finally, the complex phase velocity surface $c=c(\mathbf{n})$ is calculated using equation 25.

Inversion scheme in strong transverse isotropy

When determining parameters of strong transverse isotropy using the P-wave, we proceed in the following way: First, we readily obtain from equation 33,

$$a_{11}=c^2(\vartheta=90°), \qquad a_{33}=c^2(\vartheta=0°). \tag{37}$$

Subsequently, we evaluate parameter a_{44} and term $a_{13}(a_{13}+2a_{44})$ using the generalized linear inversion (Menke, 1989):

$$\begin{bmatrix} a_{44} \\ a_{13}(a_{13}+2a_{44}) \end{bmatrix}=\mathbf{M}^{-g}\mathbf{d}, \tag{38}$$

where $\mathbf{M}$ is the $N\times 2$ matrix, and $\mathbf{d}$ is the N-vector:

Table 1. Viscoelastic parameters. The two-index Voigt notation is used for density-normalized elastic parameters a_{ijkl}^R and quality-factor matrix q_{ijkl}.

	Elastic parameters				Attenuation parameters			
Model	a_{11}^R (km²/s²)	a_{13}^R(km²/s²)	a_{33}^R(km²/s²)	a_{44}^R(km²/s²)	Q_{11}	Q_{13}	Q_{33}	Q_{44}
1	26.54	15.49	16.23	4.41	29.7	52.8	18.2	20.3
2	14.40	4.50	9.00	2.25	30	15	20	15
3	10.80	3.53	9.00	2.25	60	30	40	30
4	10.80	3.53	9.00	2.25	240	120	160	120

Table 2. P-wave velocity and attenuation anisotropy, where $\bar{V}^{\mathrm{ray}}$, $\bar{A}^{\mathrm{ray}}$, and $\bar{Q}^{\mathrm{ray}}$ are the mean P-wave ray velocity, attenuation, and Q-factor; a_V^{ray}, a_A^{ray}, and a_Q^{ray} are the P-wave ray velocity anisotropy, attenuation anisotropy, and Q-factor anisotropy. The anisotropy is calculated as $a = 2\ (U_{\mathrm{MAX}} - U_{\mathrm{MIN}})/(U_{\mathrm{MAX}} + U_{\mathrm{MIN}})$, where U_{MAX} and U_{MIN} are the maximum and minimum values of the respective quantity.

Model	$\bar{V}^{\mathrm{ray}}$ (km/s)	a_V^{ray} (%)	$\bar{A}^{\mathrm{ray}}$ (s/km)	a_A^{ray} (%)	$\bar{Q}^{\mathrm{ray}}$	a_Q^{ray} (%)
1	4.64	24.4	46.6×10^{-4}	71.0	24.7	49.9
2	3.28	23.3	75.2×10^{-4}	67.7	21.1	48.1
3	3.06	10.5	39.5×10^{-4}	58.0	42.6	48.3
4	3.06	10.5	9.9×10^{-4}	58.0	170.5	48.3

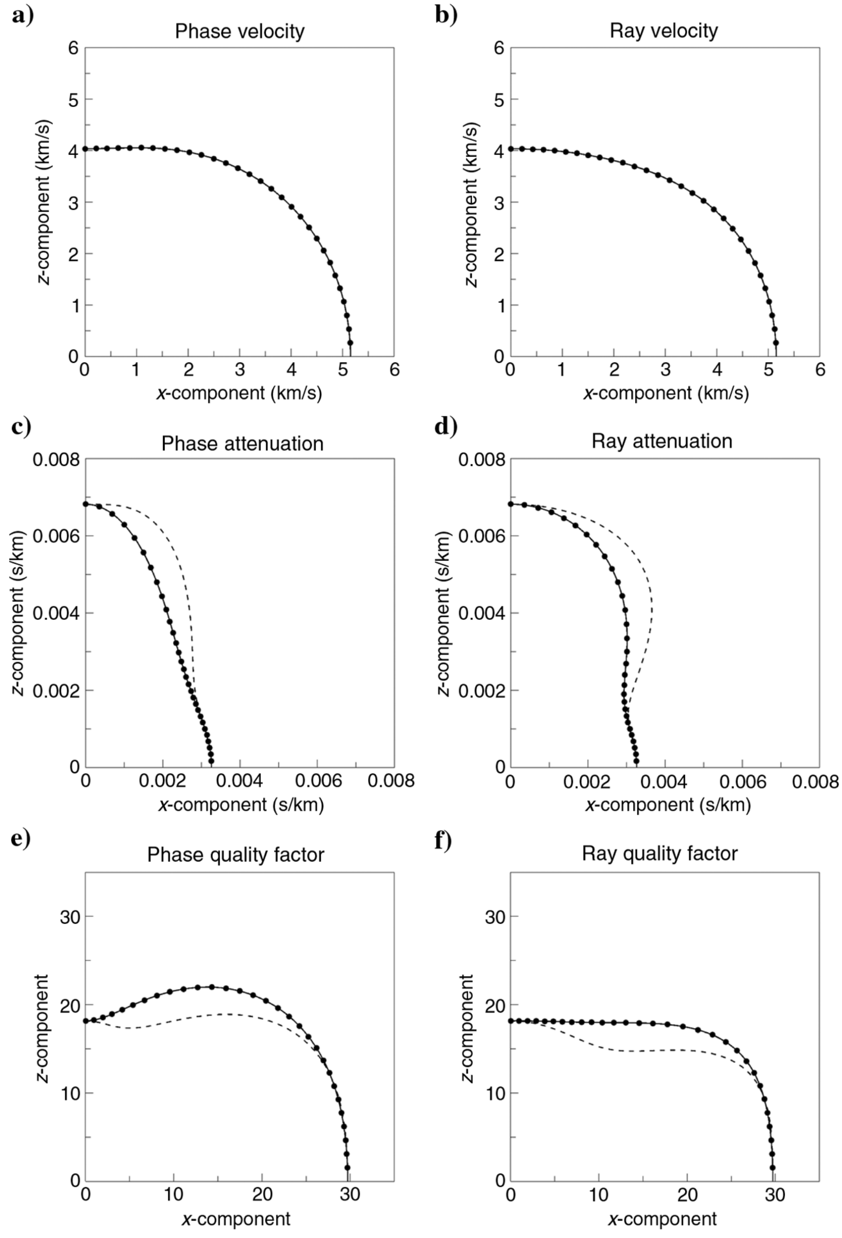

Figure 1. (a and b) Polar plots of the P-wave velocities, (c and d) attenuations, and (e and f) Q-factors in model 1. Full dots, the true quantities; solid line, the correct inversion; and dashed line, the approximate inversion. The quantities are inverted from values sampled with a step of 1°. The true quantities are displayed with a step of 3°. For the parameters of the model, see Table 1.

$$M_{i1} = a_{11}\sin^4\vartheta_i + a_{33}\cos^4\vartheta_i - c^2(\vartheta_i),$$
$$M_{i2} = -\sin^2\vartheta_i\cos^2\vartheta_i, \quad (39)$$

$$d_i = -a_{11}a_{33}\sin^2\vartheta_i\cos^2\vartheta_i + c^2(\vartheta_i)(a_{11}\sin^2\vartheta_i + a_{33}\cos^2\vartheta_i) - c^4(\vartheta_i), \quad (40)$$

and subscript i, $i = 1, \ldots, N$, is the sequential number of the measurement. Symbol $\mathbf{M}^{-g}$ in equation 38 means the generalized inverse of matrix $\mathbf{M}$. Finally, parameter a_{13} is calculated from $a_{13}(a_{13} + 2a_{44})$ as a root of the quadratic equation.

Hence, we are able to determine parameters a_{11}, a_{33}, a_{44}, and a_{13} using measurements of the P-wave in strong transverse isotropy. We cannot determine parameter a_{66}, which controls propagation of the SH-wave. For weak transverse isotropy, retrieving parameter a_{44} using equation 38 becomes unstable and measurements of the SV- or SH-waves should also be used.

Inversion scheme in weak transverse isotropy

If we assume a weakly anisotropic and attenuating medium, the complex phase velocity c can be linearized as follows (Vavryčuk, 2008, 2009):

$$c^2 = c_0^2 + \Delta c^2 = c_0^2(1 + \Delta\varepsilon), \quad (41)$$

where c_0 is the complex phase velocity in the isotropic reference medium calculated from the mean ray velocity α_0 and ray quality factor Q_0 using equation 28. Taking into account equations 26 and 33, we can write it as (Vavryčuk [2009], equation 29)

$$\Delta c^2 = \Delta a_{11}\sin^4\vartheta + \Delta a_{33}\cos^4\vartheta + 2(\Delta a_{13} + 2\Delta a_{44})\sin^2\vartheta\cos^2\vartheta, \quad (42)$$

or equivalently

$$\Delta\varepsilon = 2(\varepsilon_x\sin^4\vartheta + \varepsilon_z\cos^4\vartheta + \delta_x\sin^2\vartheta\cos^2\vartheta), \quad (43)$$

where the following complex weak anisotropy-attenuation (WAA) parameters describing the weak viscoelastic transverse isotropy were used (Vavryčuk, 2009):

$$\varepsilon_x = \frac{a_{11} - c_0^2}{2c_0^2}, \quad \varepsilon_z = \frac{a_{33} - c_0^2}{2c_0^2}, \quad \delta_x = \frac{a_{13} + 2a_{44} - c_0^2}{c_0^2}. \quad (44)$$

Finally, the WAA parameters are obtained using the following equation:

$$\begin{bmatrix}\varepsilon_x \\ \varepsilon_z \\ \delta_x\end{bmatrix} = \frac{1}{2}\begin{bmatrix}\sin^4\vartheta_1 & \cos^4\vartheta_1 & \sin^2\vartheta_1\cos^2\vartheta_1 \\ \sin^4\vartheta_2 & \cos^4\vartheta_2 & \sin^2\vartheta_2\cos^2\vartheta_2 \\ \ldots & \ldots & \ldots \\ \sin^4\vartheta_N & \cos^4\vartheta_N & \sin^2\vartheta_N\cos^2\vartheta_N\end{bmatrix}^{-g} \cdot \begin{bmatrix}\Delta\varepsilon_1 \\ \Delta\varepsilon_2 \\ \ldots \\ \Delta\varepsilon_N\end{bmatrix}, \quad (45)$$

where exponent $-g$ means the generalized inversion (Menke, 1989) and N is the number of measurements.

Table 3. Maximum errors of the correct and approximate inversions. The error for a particular ray is calculated as $E = |U^{\mathrm{exact}} - U^{\mathrm{approx}}|/U^{\mathrm{exact}}$, where U^{exact} and U^{approx} are exact and approximate values of the respective quantity. The presented values are maxima over all rays.

Model	Error V^{ray} (%)	Error A^{ray} (%)	Error Q^{ray} (%)	Error V^{ray} (%)	Error A^{ray} (%)	Error Q^{ray} (%)
	Correct inversion			Approximate inversion		
			Noise-free data, step of 1°			
1	5.6×10^{-3}	1.7×10^{-2}	1.1×10^{-2}	4.7×10^{-2}	21.0	17.3
2	2.1×10^{-2}	1.7×10^{-1}	1.5×10^{-1}	7.5×10^{-2}	25.0	20.0
3	1.1×10^{-2}	8.3×10^{-2}	7.3×10^{-2}	1.1×10^{-2}	13.7	12.1
4	1.1×10^{-2}	9.6×10^{-2}	8.4×10^{-2}	1.1×10^{-2}	13.7	12.1
			Noise-free data, step of 5°			
1	1.4×10^{-1}	4.1×10^{-1}	2.7×10^{-1}	1.4×10^{-1}	21.1	17.3
2	5.3×10^{-1}	4.1	3.5	5.3×10^{-1}	24.9	19.9
3	2.9×10^{-1}	2.3	2.0	2.9×10^{-1}	13.6	12.0
4	2.9×10^{-1}	2.3	2.0	2.9×10^{-1}	13.6	12.0
			Noisy data, step of 1°			
1	5.9×10^{-3}	1.3	1.3	4.6×10^{-2}	19.7	16.5
2	2.1×10^{-2}	1.2	1.2	6.8×10^{-2}	23.2	18.9
3	1.1×10^{-2}	1.1	1.1	1.1×10^{-2}	13.6	12.0
4	1.1×10^{-2}	1.1	1.1	1.1×10^{-2}	13.6	12.0

NUMERICAL EXAMPLES

In this section, the inversion for viscoelastic anisotropy is demonstrated on numerical examples performed for the P-wave propagating in four models of a transversely isotropic viscoelastic medium. The models display different strengths of velocity anisotropy and attenuation (see Table 1). Model 1 is taken from Carcione and Cavallini (1995) and describes properties of the clay shale. Models 2 and 3 are synthetic models taken from Vavryčuk's (2007b) models A and D with values close to observations for sedimentary rocks. Model 4 is a modification of model 3 having attenuation four times weaker. The P-wave ray velocity anisotropy of the four models is 24%, 23%, 11%, and 11%, respectively. The mean ray Q-factor of the P-wave is 25, 21, 43, and 171, respectively. The ray Q-factor anisotropy of the P-wave is approximately 50% for all models. Hence, the models cover a broad range of strengths of velocity anisotropy with various levels of attenuation (see Table 2).

Figure 1 shows directional variations in phase and ray velocities, attenuations, and Q-factors in the $x_1 - x_3$ plane for model 1. The angles ϑ (for the phase quantities) and θ (for the ray quantities) are real and range in calculations from 0° to 90° with a step of 1°. The velocities, attenuations, and Q-factors are calculated using the following three approaches.

First, the quantities are calculated from true viscoelastic parameters using forward modeling (full dots in Figure 1). The complex phase velocity c is calculated using equation 33 for real angle ϑ. Because the slowness direction is real, the complex slowness vector is homogeneous. The intrinsic phase quantities V^{phase}, A^{phase}, and

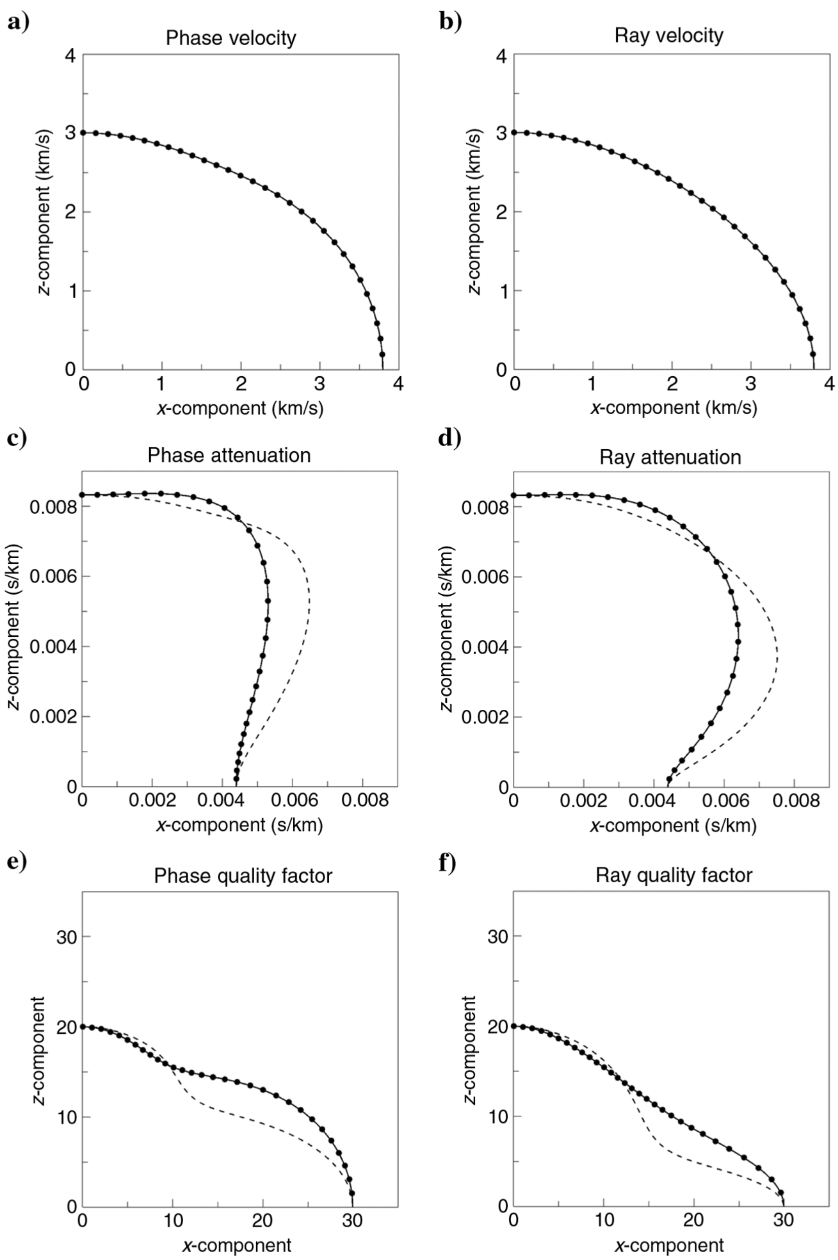

Figure 2. (a and b) Polar plots of the P-wave velocities, (c and d) attenuations, and (e and f) Q-factors in model 2. Full dots, the true quantities; solid line, the correct inversion; and dashed line, the approximate inversion. The true quantities are displayed with a step of 3°. For the parameters of the model, see Table 1.

Q^{phase} are calculated using equations 15 and 16. The ray quantities V^{ray}, A^{ray}, and Q^{ray} are calculated from the homogeneous complex energy velocity vector $\mathbf{v}$ (i.e., angle θ is real) using equations 18 and 19. However, calculating the energy velocity vector $\mathbf{v}$ is not straightforward. First, we have to calculate the corresponding stationary slowness vector $\mathbf{p}_0$ (Vavryčuk, 2007a, 2008). This vector is generally inhomogeneous (i.e., angle ϑ is complex), and it is calculated using iterations. We start with some guess of the slowness direction (e.g., with real angle ϑ), and we calculate the complex energy velocity vector $\mathbf{v}$ and its direction using equation 9. During iterations, we seek complex slowness vector $\mathbf{p}_0$ (and complex angle ϑ) for which vector $\mathbf{v}$ is homogeneous (i.e., angle θ is real) and directed along real ray direction $\mathbf{N}$. The procedure is quite similar to the shooting method in ray-tracing techniques, but it is generalized to complex algebra. Alternatively, we can calculate the stationary slowness vector $\mathbf{p}_0$ by solving the system of two algebraic equations of the fourth order in two unknown components p_1 and p_3 (Vavryčuk [2006], his equation 5.3).

Second, the quantities are inverted from true ray velocity and ray attenuation calculated in forward modeling. The inversion is performed using formulas presented in the section "Inversion scheme in strong transverse isotropy." The complex energy velocity surface is computed from the ray velocity and attenuation using equation 20. Differentiating this surface and using equation 25 for the polar reciprocity, we calculate the complex phase velocity surface $c = c(\vartheta)$, where ϑ is complex. This surface is inverted for viscoelastic parameters using equations 37–40. The predicted quantities (solid lines in Figure 1) are calculated from the retrieved parameters.

Third, the quantities are inverted from the true ray velocity and ray attenuation but the inversion is simplified and thus only approximate. The complex energy velocity surface is computed from the ray velocity and attenuation using equation 20. According to the

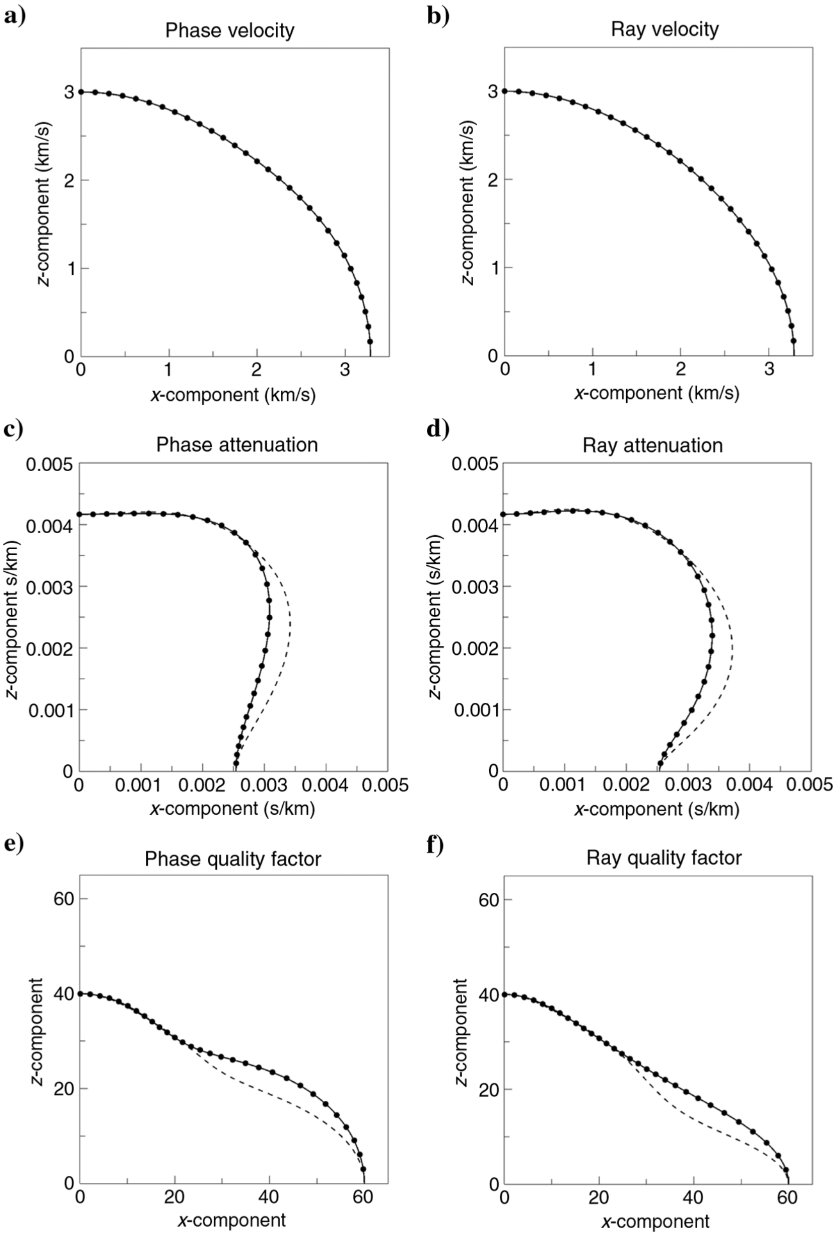

Figure 3. (a and b) Polar plots of the P-wave velocities, (c and d) attenuations, and (e and f) Q-factors in model 3. Full dots, the true quantities; solid line, the correct inversion; and dashed line, the approximate inversion. The true quantities are displayed with a step of 3°. For the parameters of the model, see Table 1.

section "Calculation of complex phase velocity," this surface is transformed to the complex phase velocity surface $c = c(\vartheta)$, where ϑ is complex and inverted for viscoelastic parameters using equations 37–40. However, the imaginary part of angle ϑ is neglected in equations 39 and 40. Hence, the slowness vector **p** is assumed to be homogeneous in the inversion. The predicted quantities (dashed lines in Figure 1) are calculated from the retrieved parameters.

Hence, the first approach yields the true quantities that serve as the reference accurate solution. The second approach demonstrates the effectiveness and robustness of the correctly performed inversion. This inversion should yield true quantities if noise-free data in a dense grid of directions are inverted. The third approach demonstrates the effectiveness and accuracy of the simplified inversion when the inhomogeneity of the slowness vector **p** is neglected and the difference between the phase and ray attenuations is partially ignored. A similar inversion is often used in weak elastic anisotropy, in which the group and phase velocities are approximately equal (Vavryčuk, 1997; Pšenčík and Vavryčuk, 2002). This approach is popular in studies of elastic anisotropy, and it produces accurate results for a broad range of real rocks.

The comparison of true and predicted quantities for model 1 (see Figure 1) shows that correct and approximate inversions yield accurate ray and phase velocities. With regard to attenuations and Q-factors, the correct inversion produces results that are almost identical with the true quantities, when the noise-free data are inverted (for errors, see Table 3). The reason for a slight discrepancy lies in the fact that the input data were sampled with a step of 1°. In the case of a denser sampling, the errors would decrease. But high-density sampling would probably be difficult to achieve in practice. The approximate inversion performs, however, much worse than the correct inversion. This result shows that the inhomogeneity of the stationary slowness direction cannot be neglected in the inversion.

The results of the inversion for models 2 and 3 are shown in Figures 2 and 3. These models display the same attenuation anisotropy, but the mean attenuation of model 3 is twice lower than that of model 2. Also, the velocity anisotropy of model 3 is twice lower than that of model 2. As a result, the accuracy of the simplified inversion is twice higher for model 3 than for model 2. Nevertheless, the errors in attenuations and Q-factors produced by the approximate inversion are still clearly visible (see Figure 3c–3f) attaining values of 13.7% and 12.1% for the ray attenuation and the ray Q-factor, respectively (see Table 3). For model 4, the phase and ray velocities are identical with those shown in Figure 3a and 3b for model 3. The phase and ray attenuations and Q-factors for model 4 are also identical with those shown in Figure 3c–3f for model 3, but the scale is four times lower for attenuations and four times higher for Q-factors. The errors of the approximate inversion are equal for models 3 and 4. Hence, decreasing the attenuation does not result in improving the accuracy of the approximate inversion.

The above inversions were performed for noise-free and densely sampled input data. The performance of the inversions for a less favorable configuration and for data with noise is summarized in Table 3 and demonstrated for model 2 in Figures 4–6. Figure 4 shows the inversion when the input data are sampled with a step of 5°. As expected, the accuracy is lower for the correct inversion but the decrease in the accuracy is not significant. The accuracy for the approximate inversion is unchanged. Figure 5 shows noisy ray velocities and ray attenuations used in the last numerical experi-

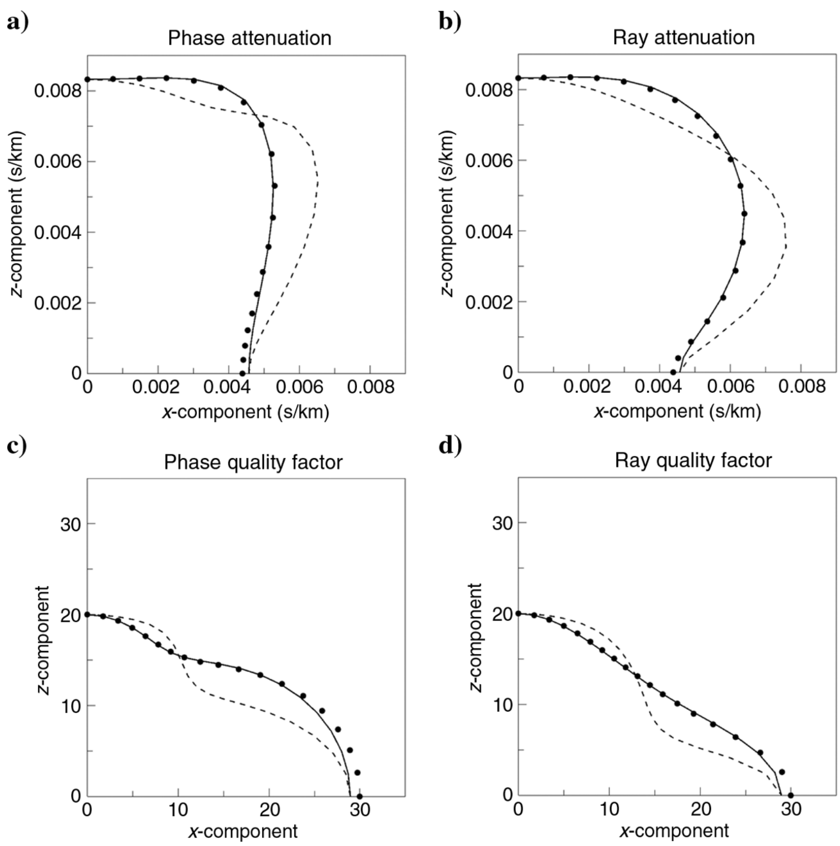

Figure 4. (a and b) Polar plots of the P-wave attenuations and (c and d) Q-factors in model 2. Full dots, the true quantities; solid line, the correct inversion; and dashed line, the approximate inversion. The quantities are inverted from values sampled with a step of 5°. For parameters of the model, see Table 1. The P-wave velocities are not displayed because they are identical to Figure 2a and 2b.

ment. The noise is random with a uniform distribution with maximum levels of ±3% and ±6% for the velocity and attenuation, respectively. The higher level of noise in attenuation reflects the fact that the wave amplitudes are usually measured with higher uncertainties than the arrival times. The noisy data were smoothed by the moving average method (solid line in Figure 5) and then inverted using the correct and approximate inversions. The results shown in Figure 6 indicate that the inversions are stable even for noisy input data. Obviously, the accuracy of the inversions would decrease for increasing noise. The key step in the inversion is smoothing the complex energy velocity surface and thus eliminating its roughness produced by noise. Because the energy velocity surface is differentiated using equation 36 before the inversion, this surface must be sufficiently smooth. Otherwise, the errors in input data are amplified by the differentiation and the inversion can fail.

DISCUSSION

In anisotropic media, we strictly distinguish between phase and ray velocities. The phase velocity describes the propagation of plane waves, whereas the ray velocity describes the propagation of a signal and energy transport. The difference between the phase and ray quantities is appropriately treated in most studies of velocity

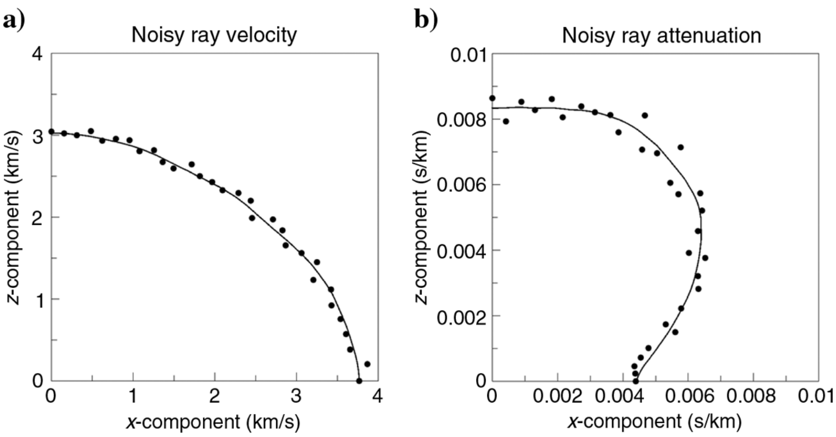

Figure 5. (a) Polar plots of the P-wave ray velocity and (b) ray attenuation in model 2 contaminated by noise. Full dots, the noisy values; solid lines, the smoothed values. The noisy values are displayed with a step of 3°. For parameters of the model, see Table 1.

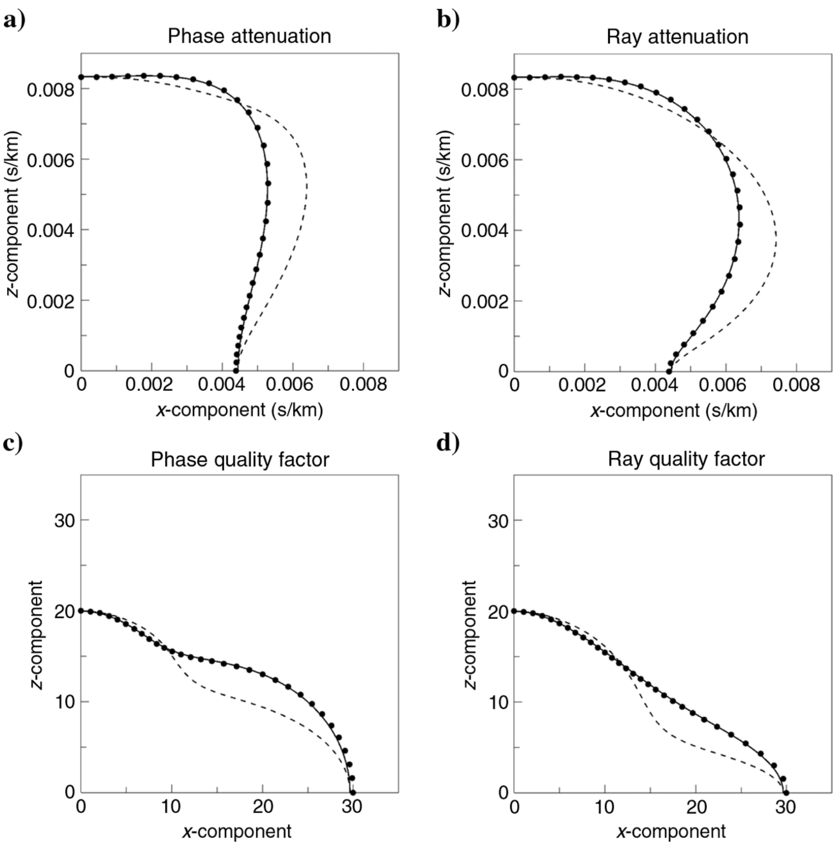

Figure 6. (a and b) Polar plots of the P-wave attenuations and (c and d) Q-factors in model 2 obtained by the inversion of noisy data. Full dots, the true quantities; solid line, the correct inversion; and dashed line, the approximate inversion. The true quantities are displayed with a step of 3°. For the parameters of the model, see Table 1. The P-wave velocities are not displayed because they are almost identical with Figure 2a and 2b.

anisotropy and can be neglected only if anisotropy is weak (Thomsen, 1986).

However, the difference between the phase and ray quantities is so far mostly ignored in studies of attenuation anisotropy. Usually, the propagation of plane waves is studied and only the phase quantities are evaluated. The inhomogeneity of the slowness vector is either assumed to be zero (Červený and Pšenčík, 2008) or taken as a free parameter. Applicability of this approach is, however, limited. Most commonly, wavefields observed in field experiments or in lab measurements are generated by point sources, and the ray velocity and attenuation are measured and used when inverting for viscoelastic parameters. Consequently, the propagating wavefronts are nonplanar and the corresponding slowness vectors are inhomogeneous.

The proper inversion producing accurate results for elastic and attenuation parameters is performed as follows: First, the complex energy velocity surface is constructed from the real ray velocity and real attenuation measured for a set of ray directions. Second, the complex slowness surface is computed using the relation of polar reciprocity between the energy velocity and slowness vectors. Although the energy velocity vectors are homogeneous, the corresponding slowness vectors are inhomogeneous. Finally, the complex phase velocity surface is calculated and inverted using the Christoffel equation.

Importantly, the applicability of weak anisotropy approximation in viscoelastic inversions is limited. This approximation works well if the velocity anisotropy as well as the attenuation anisotropy are weak. However, the assumption of weak attenuation anisotropy and of the homogeneous slowness vector is rather restrictive. Attenuation itself is not decisive for the accuracy of the approximate inversion, but the directional variations of the velocity and attenuation are essential and must be weak. This applies even to very weakly attenuating media (e.g., with the mean Q-factor 170, as in model 4, or higher). Unfortunately, the directional variation of attenuation for rocks is usually much more pronounced than that of the velocity. So, most rocks that satisfy the standard weak velocity anisotropy condition probably violate the condition of weak attenuation anisotropy. As a consequence, the differences between the ray and phase attenuations and Q-factors are not negligible and the inversion for attenuation should be performed using the correctly computed complex phase velocity. The inhomogeneity of the complex slowness vector cannot be neglected in the inversion.

ACKNOWLEDGMENTS

I thank M. van der Baan, F. Cavallini, S. Picotti, B. Shekar, and one anonymous reviewer for their helpful reviews. The study was supported by the Grant Agency of the Czech Republic (project no. P210/12/1491).

REFERENCES

Auld, B. A., 1973, Acoustic fields and waves in solids: Wiley.
Babuška, V., and M. Cara, 1991, Seismic anisotropy in the earth: Kluwer.
Backus, G. E., 1962, Long-wave elastic anisotropy produced by horizontal layering: Journal of Geophysical Research, **67**, 4427–4440, doi: 10.1029/JZ067i011p04427.
Burton, N., 2007, Rock quality, seismic velocity, attenuation and anisotropy: Taylor & Francis.
Carcione, J. M., 1990, Wave propagation in anisotropic linear viscoelastic media: Theory and simulated wavefields: Geophysical Journal International, **101**, 739–750, doi: 10.1111/j.1365-246X.1990.tb05580.x.
Carcione, J. M., 1993, Seismic modeling in viscoelastic media: Geophysics, **58**, 110–120, doi: 10.1190/1.1443340.
Carcione, J. M., 1994, Wavefronts in dissipative anisotropic media: Geophysics, **59**, 644–657, doi: 10.1190/1.1443624.
Carcione, J. M., 2000, A model for seismic velocity and attenuation in petroleum source rocks: Geophysics, **65**, 1080–1092, doi: 10.1190/1.1444801.
Carcione, J. M., 2007, Wave fields in real media: Theory and numerical simulation of wave propagation in anisotropic, anelastic, porous and electromagnetic media: Elsevier.
Carcione, J. M., and F. Cavallini, 1993, Energy balance and fundamental relations in anisotropic-viscoelastic media: Wave Motion, **18**, 11–20, doi: 10.1016/0165-2125(93)90057-M.
Carcione, J. M., and F. Cavallini, 1995, Attenuation and quality factor surfaces in anisotropic-viscoelastic media: Mechanics of Materials, **19**, 311–327, doi: 10.1016/0167-6636(94)00040-N.
Carcione, J. M., G. Quiroga-Goode, and F. Cavallini, 1996, Wavefronts in dissipative anisotropic media: Comparison of the plane wave theory with numerical modeling: Geophysics, **61**, 857–861, doi: 10.1190/1.1444010.
Červený, V., and I. Pšenčík, 2005, Plane waves in viscoelastic anisotropic media: Part I — Theory: Geophysical Journal International, **161**, 197–212, doi: 10.1111/j.1365-246X.2005.02589.x.
Červený, V., and I. Pšenčík, 2008, Quality factor Q in dissipative anisotropic media: Geophysics, **73**, no. 4, T63–T75, doi: 10.1190/1.2937173.
Chichinina, T., V. Sabinin, and G. Ronquillo-Jarillo, 2006, QVOA analysis: P-wave attenuation anisotropy for fracture characterization: Geophysics, **71**, no. 3, C37–C48, doi: 10.1190/1.2194531.
Deschamps, M., and F. Assouline, 2000, Attenuation along the Poynting vector direction of inhomogeneous plane waves in absorbing and anisotropic solids: Acustica, **86**, 295–302.
Deschamps, M., B. Poirée, and O. Poncelet, 1997, Energy velocity of complex harmonic plane waves in viscous fluids: Wave Motion, **25**, 51–60, doi: 10.1016/S0165-2125(96)00032-7.
Farra, V., and I. Pšenčík, 2008, First-order ray computations of coupled S waves in inhomogeneous weakly anisotropic media: Geophysical Journal International, **173**, 979–989, doi: 10.1111/j.1365-246X.2008.03778.x.
Gajewski, D., and I. Pšenčík, 1992, Vector wavefields for weakly attenuating anisotropic media by the ray method: Geophysics, **57**, 27–38, doi: 10.1190/1.1443186.
Grechka, V., 2015, Shear-wave group-velocity surfaces in low-symmetry anisotropic media: Geophysics, **80**, no. 1, C1–C7, doi: 10.1190/GEO2014-0156.1.
Hearn, D. J., and E. S. Krebes, 1990, On computing ray-synthetic seismograms for anelastic media using complex rays: Geophysics, **55**, 422–432, doi: 10.1190/1.1442851.
Helbig, K., 1994, Foundations of anisotropy for exploration seismics: Pergamon.
Hudson, J. A., 1981, Wave speeds and attenuation of elastic waves in material containing cracks: Geophysical Journal of the Royal Astronomical Society, **64**, 133–150, doi: 10.1111/j.1365-246X.1981.tb02662.x.
Karato, S., 2008, An introduction to rheology of solid earth: Cambridge University Press.
Lipschutz, S., 1969, Theory and problems of differential geometry: McGraw-Hill.
Menke, W., 1989, Ceophysical data analysis: Discrete inverse theory: Academic Press.
Mensch, T., and P. Rasolofosaon, 1997, Elastic-wave velocities in anisotropic media of arbitrary symmetry — Generalization of Thomsen's parameters ε, δ and γ: Geophysical Journal International, **128**, 43–64, doi: 10.1111/j.1365-246X.1997.tb04070.x.
Picotti, S., J. M. Carcione, J. E. Santos, and D. Gei, 2010, Q-anisotropy in finely-layered media: Geophysical Research Letters, **37**, L06302, doi: 10.1029/2009GL042046.
Pšenčík, I., and D. Gajewski, 1998, Polarization, phase velocity and NMO velocity of qP waves in arbitrary weakly anisotropic media: Geophysics, **63**, 1754–1766, doi: 10.1190/1.1444470.
Pšenčík, I., and V. Vavryčuk, 2002, Approximate relation between the ray vector and wave normal in weakly anisotropic media: Studia Geophysica et Geodaetica, **46**, 793–807, doi: 10.1023/A:1021189724526.
Rasolofosaon, P., 2010, Generalized anisotropy parameters and approximations of attenuations and velocities in viscoelastic media of arbitrary anisotropy type — Theoretical and experimental aspects: Geophysical Prospecting, **58**, 637–655, doi: 10.1111/j.1365-2478.2009.00863.x.
Schoenberg, M., and J. Douma, 1988, Elastic wave propagation in media with parallel fractures and aligned cracks: Geophysical Prospecting, **36**, 571–590, doi: 10.1111/j.1365-2478.1988.tb02181.x.
Shuvalov, A. L., and N. H. Scott, 1999, On the properties of homogeneous viscoelastic waves: Quarterly Journal of Mechanics and Applied Mathematics, **52**, 405–417., doi: 10.1093/qjmam/52.3.405.
Svitek, T., V. Vavryčuk, T. Lokajíček, and M. Petružálek, 2014, Determination of elastic anisotropy of rocks from P- and S-wave velocities: Numeri-

cal modelling and lab measurements: Geophysical Journal International, **199**, 1682–1697, doi: 10.1093/gji/ggu332.

Thomsen, L., 1986, Weak elastic anisotropy: Geophysics, **51**, 1954–1966, doi: 10.1190/1.1442051.

Vavryčuk, V., 1997, Elastodynamic and elastostatic Green tensors for homogeneous weak transversely isotropic media: Geophysical Journal International, **130**, 786–800, doi: 10.1111/j.1365-246X.1997.tb01873.x.

Vavryčuk, V., 2006, Calculation of the slowness vector from the ray vector in anisotropic media: Proceedings of the Royal Society A, **462**, 883–896, doi: 10.1098/rspa.2005.1605.

Vavryčuk, V., 2007a, Asymptotic Green's function in homogeneous anisotropic viscoelastic media: Proceedings of the Royal Society, A, **463**, 2689–2707, doi: 10.1098/rspa.2007.1862.

Vavryčuk, V., 2007b, Ray velocity and ray attenuation in homogeneous anisotropic viscoelastic media: Geophysics, **72**, no. 6, D119–D127, doi: 10.1190/1.2768402.

Vavryčuk, V., 2008, Velocity, attenuation and quality factor in anisotropic viscoelastic media: A perturbation approach: Geophysics, **73**, no. 5, D63–D73, doi: 10.1190/1.2921778.

Vavryčuk, V., 2009, Weak anisotropy-attenuation parameters: Geophysics, **74**, no. 5, WB203–WB213, doi: 10.1190/1.3173154.

Vavryčuk, V., 2012, On numerically solving the complex Eikonal equation using real ray-tracing methods: A comparison with the exact analytical solution: Geophysics, **77**, no. 4, T109–T116, doi: 10.1190/geo2011-0431.1.

Zhu, T. F., and K. Y. Chun, 1994, Complex rays in elastic and anelastic media: Geophysical Journal International, **119**, 269–276, doi: 10.1111/j.1365-246X.1994.tb00927.x.

Zhu, Y., and I. Tsvankin, 2006, Plane-wave propagation in attenuative transversely isotropic media: Geophysics, **71**, no. 2, T17–T30, doi: 10.1190/1.2187792.

Zhu, Y., and I. Tsvankin, 2007, Plane-wave attenuation anisotropy in orthorhombic media: Geophysics, **72**, no. 1, D9–D19, doi: 10.1190/1.2387137.