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Multipolar elastic fields in homogeneous isotropic media by higher-order ray approximations

V. Vavryčuk* and K. Yomogida

Department of Earth and Planetary Systems Science, Faculty of Science, Hiroshima University, 1-3-1 Kagami-yama, Higashi-Hiroshima 739, Japan

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SUMMARY

An analytical approach to the calculation of higher-order ray approximations for wavefields generated by point sources in homogeneous isotropic media is presented. It is shown that the near-field waves neglected by the zeroth-order ray approximation can be incorporated by considering higher-order terms of the ray series. Assuming the point source to be a unit single force, we calculate the complete ray-theoretical Green's function. The ray series of the Green's function consists only of three non-zero terms and the ray solution coincides with an exact solution. Wavefields radiated by general multipolar sources are also exactly expressed by the ray theory. Simple algebraic formulae for the spherical harmonics coefficients of all higher-order ray approximations are established.

Key words: Green's function, higher-order ray approximations, multipolar sources, near-field waves, singular wavefield.

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1 INTRODUCTION

Ray theory was first developed for solving the elastodynamic equation by Babich (1956), Babich & Alekseyev (1958) and Karal & Keller (1959). The ray series includes, in general, an infinite number of higher-order terms, but only the leading (zeroth-order) term is in standard use. The higher-order terms are usually neglected, since their calculations are more complicated and in most cases their contribution is rather weak. Therefore, including the higher-order terms in numerical calculations is still rare, even in problems for which they cannot be neglected. Thus the accuracy of results may be considerably distorted. The effects produced by the higher-order terms are often referred as 'non-geometrical' to emphasize that they cannot be explained by the classical ray theory known from geometrical optics. A frequently discussed problem studied by many authors numerically as well as analytically (e.g. Daley & Hron 1987; Roslov & Yanovskaya 1988; Kiselev & Roslov 1991; Hron & Zheng 1993; Santos & Pšenčík 1993) is the reflection of a spherical P wave at a free surface or at a plane interface. For this case, remarkable effects caused by the higher-order ray approximations are reported, particularly for PS reflections.

In this study, we shall take a different approach to the higher-order ray approximations. We consider a simple problem: a spherical wave propagating in an elastic, homogeneous and isotropic medium. Since we deal only with a spherical wave, we are able to solve the basic equations of ray theory analytically and to give explicit formulae for the higher-order terms. Moreover, the exact solution for this problem is well known, so we can compare the ray-theoretical formulae with the exact ones. In particular, we will examine the relation between the higher-order ray approximations and the near-field waves, which are present in exact solutions but intractable by the zeroth-order ray approximation. First, we will study the ray-theoretical Green's function, and then wavefields radiated by multipolar sources. We will show that the near-field waves are *perfectly* described by the higher-order ray approximations and that the ray-theoretical Green's function coincides with an *exact* solution. Likewise, for the multipolar sources the ray theory gives the exact solution. We will show that the higher-order ray approximations can easily be computed using a spherical harmonics expansion of the ray amplitudes. Simple algebraic formulae for the spherical harmonics coefficients of the higher-order terms from coefficients of the leading term will be derived.

* On leave from: Geophysical Institute, Czech Academy of Sciences, Boční II/1401, 141 31 Praha 4, Czech Republic.

2 HIGHER-ORDER APPROXIMATIONS OF RAY THEORY

2.1 Basic equations of ray theory

For a perfectly elastic, homogeneous, isotropic, and unbounded medium without any external forces, the elastodynamic equation is

$$\rho \ddot{u}_i - (\lambda + \mu) u_{j,ji} - \mu u_{i,jj} = 0, \tag{1}$$

where $\mathbf{u}(\mathbf{x}, t)$ is the displacement vector, λ and μ are the Lamé constants and ρ is the density of the medium. Dots over quantities mean the time derivative; indices after the comma denote the spatial derivative. The Einstein summation convention of pairs of equal indices is applied. We seek a solution in the form of a ray series (Červený, Molotkov & Pšenčík 1977):

$$u_i(\mathbf{x}, t) = \sum_{K=0}^{\infty} U_i^{(K)}(\mathbf{x}) f^{(K)}(t - \tau(\mathbf{x})), \tag{2}$$

where

$$\frac{d}{dt} f^{(K)}(t) = f^{(K-1)}(t).$$

K denotes the order of the ray approximation, $\mathbf{U}^{(K)}(\mathbf{x})$ is the ray-amplitude vector and $\tau(\mathbf{x})$ is the traveltime. Inserting (2) into (1) leads to the following equations for the ray amplitudes $\mathbf{U}^{(K)}$:

$$N_i(U_k^{(K)}) - M_i(U_k^{(K-1)}) + L_i(U_k^{(K-2)}) = 0, \tag{3}$$

which are called the *basic equations of ray theory*. Differential vector operators, $\mathbf{N}, \mathbf{M}$ and $\mathbf{L}$ are defined as follows:

$$\begin{aligned}
N_i(U_j) &= -\rho U_i + (\lambda + \mu) p_i p_j U_j + \mu p_j p_j U_i, \\
M_i(U_j) &= (\lambda + \mu)\left[p_j \frac{\partial U_j}{\partial x_i} + p_i \frac{\partial U_j}{\partial x_j} + U_j \frac{\partial p_i}{\partial x_j}\right] \\
&\quad + \mu\left[2 p_j \frac{\partial U_i}{\partial x_j} + U_i \frac{\partial p_j}{\partial x_j}\right], \\
L_i(U_j) &= (\lambda + \mu) \frac{\partial^2 U_j}{\partial x_i \partial x_j} + \mu \frac{\partial^2 U_i}{\partial x_j \partial x_j},
\end{aligned} \tag{4}$$

where $p_i = \partial\tau/\partial x_i$ is the slowness vector. The basic ray-theoretical equations (3) are recursive equations: each term of the ray series is calculated from the lower-order terms. The leading term (i.e. the zeroth-order ray approximation) is calculated by assuming terms of negative order to be equal to zero ($\mathbf{U}^{(K)} = \mathbf{0}$ for $K < 0$). Consequently, the first-order term and then the other higher-order terms of the ray series can be determined. For this determination, it is convenient to introduce the so-called *additional* and *principal components* $\mathbf{U}^{(K)\perp}$ and $\mathbf{U}^{(K)\parallel}$ of the ray amplitude $\mathbf{U}^{(K)}$ as follows:

$$U_i^{P(K)} = U_i^{P(K)\parallel} + U_i^{P(K)\perp}, \qquad U_i^{S(K)} = U_i^{S(K)\parallel} + U_i^{S(K)\perp}, \tag{5a}$$

where

$$\begin{aligned}
U_i^{P(K)\parallel} &= U^{P(K)\parallel} g_i^P, \\
U_i^{P(K)\perp} &= U^{P(K)\perp SV} g_i^{SV} + U^{P(K)\perp SH} g_i^{SH}, \\
U_i^{S(K)\parallel} &= U^{SV(K)\parallel} g_i^{SV} + U^{SH(K)\parallel} g_i^{SH}, \qquad U_i^{S(K)\perp} = U^{S(K)\perp} g_i^P.
\end{aligned} \tag{5b}$$

Vectors $\mathbf{g}^P$, $\mathbf{g}^{SV}$ and $\mathbf{g}^{SH}$ are called the *polarization vectors*. They form an orthogonal system of unit vectors: $\mathbf{g}^P$ is parallel to the slowness vector $\mathbf{p}$; $\mathbf{g}^{SV}$ and $\mathbf{g}^{SH}$ are perpendicular to $\mathbf{p}$. For point sources in homogeneous isotropic media we will choose $\mathbf{g}^P$, $\mathbf{g}^{SV}$ and $\mathbf{g}^{SH}$ in the following way:

$$\mathbf{g}^P = \begin{bmatrix} \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \\ \cos\vartheta \end{bmatrix}, \qquad \mathbf{g}^{SV} = \begin{bmatrix} \cos\vartheta\cos\varphi \\ \cos\vartheta\sin\varphi \\ -\sin\vartheta \end{bmatrix}, \qquad \mathbf{g}^{SH} = \begin{bmatrix} -\sin\varphi \\ \cos\varphi \\ 0 \end{bmatrix}, \tag{6}$$

where angles ϑ and φ are the standard take-off angles of a ray.

The determination of each term of the ray series is performed in two steps: first, the additional component $\mathbf{U}^{(K)\perp}$ is calculated by the differentiation of the lower-order terms, and second, the principal component $\mathbf{U}^{(K)\parallel}$ is calculated by solving an ordinary differential equation called the *transport equation*.

2.2 Additional components

Solving eqs (3), we obtain the formulae for additional components of the higher-order ray approximations (Červený *et al.* 1977, eqs 2.16, 2.19):

$$\begin{aligned}
U_i^{P(K)\perp} &= -\frac{\alpha^2}{\lambda + \mu}(g_i^{SV} g_k^{SV} + g_i^{SH} g_k^{SH}) \\
&\quad \times \{M_k(U_j^{P(K-1)}) - L_k(U_j^{P(K-2)})\},
\end{aligned} \tag{7a}$$

for the P wave, and

$$U_i^{S(K)\perp} = \frac{\beta^2 g_i^P g_k^P}{\lambda + \mu}\{M_k(U_j^{S(K-1)}) - L_k(U_j^{S(K-2)})\}, \tag{7b}$$

for the S wave.

Obviously, the additional components of the zeroth-order terms $\mathbf{U}^{P(0)\perp}$ and $\mathbf{U}^{S(0)\perp}$ equal zero, and thus the zeroth-order term consists only of principal components.

2.3 Principal components

The principal components $\mathbf{U}^{P(K)\parallel}$ and $\mathbf{U}^{S(K)\parallel}$ can be found by solving the transport equations (Červený *et al.* 1977, eqs 2.24, 2.26). Specifying them for a homogeneous isotropic medium, we obtain

$$\frac{dU^{P(K)\parallel}}{d\tau} + \frac{U^{P(K)\parallel}}{2J}\frac{dJ}{d\tau} = \frac{1}{2\rho}\{L_n(U_k^{P(K-1)}) - M_n(U_k^{P(K)\perp})\} g_n^P, \tag{8a}$$

$$\frac{dU^{SV(K)\parallel}}{d\tau} + \frac{U^{SV(K)\parallel}}{2J}\frac{dJ}{d\tau} = \frac{1}{2\rho}\{L_n(U_k^{S(K-1)}) - M_n(U_k^{S(K)\perp})\} g_n^{SV}, \tag{8b}$$

$$\frac{dU^{SH(K)\parallel}}{d\tau} + \frac{U^{SH(K)\parallel}}{2J}\frac{dJ}{d\tau} = \frac{1}{2\rho}\{L_n(U_k^{S(K-1)}) - M_n(U_k^{S(K)\perp})\} g_n^{SH}. \tag{8c}$$

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Since the equations for S waves (8b, c) are essentially the same as for P waves (8a), we will consider only P waves in the following derivation. For S waves we will present just final formulae.

Assuming waves to be generated by a point source located at the origin of coordinates, the wavefront becomes spherical and the ray coordinates coincide with the spherical coordinates ϑ, φ, r. Inserting the ray Jacobian J,

$$J = \left|\frac{\partial(x_1, x_2, x_3)}{\partial(\vartheta, \varphi, r)}\right| = r^2 \sin\vartheta, \tag{9}$$

into (8a) and using as the variable the distance r from the source to a receiver instead of the traveltime τ, the transport equation for P waves reads

$$\frac{dU^{P(K)\|}}{dr} + \frac{U^{P(K)\|}}{r} = \frac{1}{2\rho\alpha}\{L_n(U_k^{P(K-1)}) - M_n(U_k^{P(K)\perp})\}g_n^P. \tag{10}$$

Eq. (10) is an ordinary differential inhomogeneous equation of the first order. The zeroth-order ray approximation is a solution of the homogeneous equation

$$\frac{dU^{P(0)\|}}{dr} + \frac{U^{P(0)\|}}{r} = 0, \tag{11}$$

taking the form

$$U^{P(0)\|} = \frac{\mathscr{R}^{P(0)\|}(\vartheta, \varphi)}{r}. \tag{12}$$

$\mathscr{R}^{P(0)\|}(\vartheta, \varphi)$ denotes the radiation pattern of the P waves in the zeroth-order ray approximation. The form of this function is not predicted by the ray theory and depends on the type of source. In fact, this function represents the far-field radiation pattern of a source and we will assume it to be known before calculations of the higher-order ray approximations. Inserting (12) into (7a) and (10), we find the transport equation for the first-order ray approximation:

$$\frac{dU^{P(1)\|}}{dr} + \frac{U^{P(1)\|}}{r} = \frac{f^P(\vartheta, \varphi)}{r^3}, \tag{13}$$

where $f^P(\vartheta, \varphi)$ depends on the form of $\mathscr{R}^{P(0)\|}(\vartheta, \varphi)$. The solution can formally be written as

$$U^{P(1)\|} = \frac{\mathscr{R}^{P(1)\|}(\vartheta, \varphi)}{r^2} + \frac{C^{P(1)}(\vartheta, \varphi)}{r}, \tag{14}$$

where $\mathscr{R}^{P(1)\|}(\vartheta, \varphi)$ is the radiation pattern of the principal P-wave component in the first-order ray approximation. The function $C^{P(1)}(\vartheta, \varphi)$ is an arbitrary integration constant, since a general solution should involve any solution of the homogeneous equation. It implies that any arbitrarily chosen far-field term in (14) is also a solution of (13). As we mentioned, we will assume the far-field term to be known and it will be identified with the zeroth-order ray approximation. Therefore, no other far-field terms can be included in solution (14), and thus the integration constant $C^{P(1)}(\vartheta, \varphi)$ is identically zero. Analogously, we can write for the Kth-order ray approximation

$$U^{P(K)\|} = \frac{\mathscr{R}^{P(K)\|}(\vartheta, \varphi)}{r^{K+1}} + \frac{C^{P(K)}(\vartheta, \varphi)}{r}, \tag{15}$$

and we can again conclude that the integration constant $C^{P(K)}(\vartheta, \varphi)$ is zero. Finally, we arrive at an explicit formula for the principal component of the Kth-order ray approximation for P waves ($K > 0$):

$$U_i^{P(K)\|} = \frac{\alpha r g_i^P g_k^P}{2K(\lambda + 2\mu)}\{M_k(U_i^{P(K)\perp}) - L_k(U_i^{P(K-1)})\}. \tag{16a}$$

Similarly, we find for S waves that

$$U_i^{S(K)\|} = \frac{\beta r}{2K\mu}(g_i^{SV} g_k^{SV} + g_i^{SH} g_k^{SH}) \times\{M_k(U_i^{S(K)\perp}) - L_k(U_i^{S(K-1)})\}. \tag{16b}$$

From (7a,b) and (16a,b) we can see that all the higher-order ray approximations can be obtained only by the differentiation of the zeroth-order term, which is a mathematically elementary procedure. It should be noted, however, that this procedure is numerically unstable and analytically tends to produce rather extensive formulae.

The vector equations (3) can also be generalized to tensor fields which are represented, for example, by the Green's function. In this case, the ray-amplitude vector U_i is replaced by the ray-amplitude tensor U_{ij}, and vector differential operators N_i, M_i and L_i defined in (4) become tensor operators N_{ij}, M_{ij} and L_{ij}. Since the reformulation of eqs (2), (3) and (4), and consequently (7a,b) and (16a,b), to tensor notation is simple, we will not give it explicitly, and we will refer to the vector equations even in the case when we will actually calculate tensor fields.

3 GREEN'S FUNCTION FOR ISOTROPIC MEDIA

The elastodynamic Green's function for homogeneous, isotropic and unbounded media was first found by Stokes in 1849 and Love in 1903. A detailed derivation using Lamé's potentials is given, for example, in Aki & Richards (1980) and in Ben-Menahem & Singh (1981). The Green's function consists of three terms: the far-field P wave, the far-field S wave and the near-field wave. The amplitude of the far-field waves decreases with distance as $1/r$, while the amplitude of the near-field waves decreases faster. The near- and far-field waves differ also in waveforms: the waveform in the far field reproduces the source-time function, but in the near field it is more complicated, containing more frequencies that are lower.

The geometrical ray method can be used to compute a solution in the far field only. The near-field wave is a typical wave that is intractable by this method. Next, we will show, however, that by taking the higher-order ray approximations into account, the near-field wave can be also reproduced. Moreover, we show that even the exact elastodynamic Green's function can be obtained by the higher-order ray theory.

3.1 *P* wave

The far-field P wave of the Green's function is (see Aki & Richards 1980, eq. 4.24)

$$G_{kl}^{P(0)}(\mathbf{x}, t) = \frac{1}{4\pi\rho\alpha^2}\frac{g_k^P g_l^P}{r}\,\delta\left(t - \frac{r}{\alpha}\right), \tag{17}$$

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where $\delta(t)$ denotes the Dirac delta function, r is the distance of an observation point from the source and α is the P-wave velocity. Formula (17) can easily be recognized to be the zeroth-order ray approximation (eq. 12) with the ray amplitude

$$U_{kl}^{P(0)}(\mathbf{x}) = \frac{1}{4\pi\rho\alpha^2}\frac{g_k^P g_l^P}{r}. \tag{18}$$

Inserting (18) into (7a) and (16a) we can calculate the first-order ray approximation and then each higher-order term recurrently. Following this procedure, we obtain, for the first-order ray approximation,

$$U_{kl}^{P(1)\perp}(\mathbf{x}) = -\frac{1}{4\pi\rho\alpha}\frac{g_k^{SV} g_l^{SV} + g_k^{SH} g_l^{SH}}{r^2},$$

$$U_{kl}^{P(1)\parallel}(\mathbf{x}) = \frac{1}{4\pi\rho\alpha}\frac{2g_k^P g_l^P}{r^2}.$$

Then we finally obtain

$$G_{kl}^{P(1)}(\mathbf{x}, t) = \frac{1}{4\pi\rho\alpha}\frac{3g_k^P g_l^P - \delta_{kl}}{r^2} H\left(t - \frac{r}{\alpha}\right), \tag{19}$$

where $H(t)$ denotes the Heaviside step function and δ_{kl} is the Kronecker delta. In (19) we use the identity

$$g_k^P g_l^P + g_k^{SV} g_l^{SV} + g_k^{SH} g_l^{SH} = \delta_{kl}. \tag{20}$$

Consequently, for the second-order ray approximation we have

$$U_{kl}^{P(2)\perp}(\mathbf{x}) = -\frac{1}{4\pi\rho}\frac{g_k^{SV} g_l^{SV} + g_k^{SH} g_l^{SH}}{r^3},$$

$$U_{kl}^{P(2)\parallel}(\mathbf{x}) = \frac{1}{4\pi\rho}\frac{2g_k^P g_l^P}{r^3},$$

which leads to

$$G_{kl}^{P(2)}(\mathbf{x}, t) = \frac{1}{4\pi\rho}\frac{3g_k^P g_l^P - \delta_{kl}}{r^3}\int H\left(t - \frac{r}{\alpha}\right) dt. \tag{21}$$

Calculating the third- and higher-order ray approximations, we arrive at the interesting result that these approximations equal zero.

3.2 S wave

Analogously to the P wave, we can apply the above procedure to the S wave. The far-field S wave is (Aki & Richards 1980, eq. 4.25)

$$G_{kl}^{S(0)}(\mathbf{x}, t) = \frac{1}{4\pi\rho\beta^2}\frac{g_k^{SV} g_l^{SV} + g_k^{SH} g_l^{SH}}{r}\delta\left(t - \frac{r}{\beta}\right), \tag{22}$$

where β is the S-wave velocity. For the first-order ray approximation we obtain

$$U_{kl}^{S(1)\perp}(\mathbf{x}) = -\frac{1}{4\pi\rho\beta}\frac{2g_k^P g_l^P}{r^2},$$

$$U_{kl}^{S(1)\parallel}(\mathbf{x}) = \frac{1}{4\pi\rho\beta}\frac{g_k^{SV} g_l^{SV} + g_k^{SH} g_l^{SH}}{r^2},$$

then we get

$$G_{kl}^{S(1)}(\mathbf{x}, t) = -\frac{1}{4\pi\rho\beta}\frac{3g_k^P g_l^P - \delta_{kl}}{r^2} H\left(t - \frac{r}{\beta}\right), \tag{23}$$

and for the second-order ray approximation

$$U_{kl}^{S(2)\perp}(\mathbf{x}) = -\frac{1}{4\pi\rho}\frac{2g_k^P g_l^P}{r^3},$$

$$U_{kl}^{S(2)\parallel}(\mathbf{x}) = \frac{1}{4\pi\rho}\frac{g_k^{SV} g_l^{SV} + g_k^{SH} g_l^{SH}}{r^3},$$

which leads to

$$G_{kl}^{S(2)}(\mathbf{x}, t) = -\frac{1}{4\pi\rho}\frac{3g_k^P g_l^P - \delta_{kl}}{r^3}\int H\left(t - \frac{r}{\beta}\right) dt. \tag{24}$$

Similarly to the P wave, all the other higher-order terms equal zero.

3.3 Complete Green's function

Summing eqs (17), (19), (21) for the P wave and (22), (23) and (24) for the S wave, we have

$$\begin{aligned} G_{kl}(\mathbf{x}, t) = G_{kl}^P(\mathbf{x}, t) + G_{kl}^S(\mathbf{x}, t) &= \frac{1}{4\pi\rho\alpha^2}\frac{g_k^P g_l^P}{r}\delta\left(t - \frac{r}{\alpha}\right) \\ &+ \frac{1}{4\pi\rho\beta^2}\frac{g_k^{SV} g_l^{SV} + g_k^{SH} g_l^{SH}}{r}\delta\left(t - \frac{r}{\beta}\right) \\ &+ \frac{1}{4\pi\rho}\frac{3g_k^P g_l^P - \delta_{kl}}{r^3}\int_{r/\alpha}^{r/\beta}\tau\delta(t - \tau)\, d\tau, \end{aligned} \tag{25}$$

where we used

$$\begin{aligned} &\int H\left(t - \frac{r}{\alpha}\right) dt - \int H\left(t - \frac{r}{\beta}\right) dt + \frac{r}{\alpha}H\left(t - \frac{r}{\alpha}\right) - \frac{r}{\beta}H\left(t - \frac{r}{\beta}\right) \\ &= t\left[H\left(t - \frac{r}{\alpha}\right) - H\left(t - \frac{r}{\beta}\right)\right] = \int_{r/\alpha}^{r/\beta}\tau\delta(t - \tau)\, d\tau. \end{aligned} \tag{26}$$

Formula (25) expresses the complete ray-theoretical Green's function and fully coincides with an exact formula (see Aki & Richards 1980, eq. 4.23). It has, for both P and S waves, only three non-zero terms. The zeroth-order term corresponds to the far-field wave, the first- and second-order terms correspond to the near-field wave. The near-field wave in (25) is expressed only by one integral, and thus it is not obvious how to separate it formally into P and S waves. Therefore, it is not well known that the Green's function has the form of a ray series with a finite number of terms. To the authors' knowledge, this fact is explicitly reported only by Goldin & Ashkarin (1991).

Comparing (19) and (21) with (23) and (24), we can see that the formulae for the first- and second-order ray approximations for P and S waves are very similar. For the second-order ray approximation, the ray amplitudes for P and S waves are even the same, but with a reverse polarity. This fact has a well-founded physical reason: the first-order term for P and S waves considered separately produces a static offset, and the second-order term a divergence in time, but their mutual combination leads to the cancelling of these unphysical effects. The final solution does not diverge and it

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is free from the static offset. The mutual cancellation of P and S motions in the near field is also reported by Wu & Ben-Menahem (1985). Convolving the complete ray-theoretical Green's function with the unit constant in time, we obtain the elastostatic Green's function G^{Stat}_{kl} in the following form (Hirth & Lothe 1968, eq. 2-79; Mura 1982, eq. 5.8):

$$G^{\mathrm{Stat}}_{kl}=\frac{1}{8\pi\mu(\lambda+2\mu)}\frac{1}{r}[(\lambda+\mu)g^P_k g^P_l+(\lambda+3\mu)\delta_{kl}], \quad (27)$$

where we used $\alpha^2=(\lambda+2\mu)/\rho$ and $\beta^2=\mu/\rho$. Thus, by calculating the higher-order terms of the ray series from the zeroth-order ray approximation, we can even obtain an exact solution for the *static case.*

4 MULTIPOLAR FIELDS FROM POINT SOURCES

In this section, we will study waves generated by multipolar point sources. By the *multipolar point source* we mean a source generating a displacement field in the form

$$u_i(\mathbf{x},t)=G_{ik_1,k_2k_3\cdots k_N}*M_{k_1k_2k_3\ldots k_N}, \quad (28)$$

where $G_{ij}(\mathbf{x},t)$ is the Green's tensor, $M_{k_1k_2k_3\cdots k_N}(t)$ is the multipolar moment tensor, N is the order of a multipolar source and $*$ denotes the time convolution operator. For $N=1$, formula (28) reduces to

$$u_i(\mathbf{x},t)=G_{ij}*f_j, \quad (29a)$$

and for $N=2$, to

$$u_i(\mathbf{x},t)=G_{ij,k}*M_{jk}, \quad (29b)$$

where $f_j(t)$ is the single force vector and $M_{jk}(t)$ is the second-order moment tensor. As in Section 3, we will assume that we know the far-field waves generated by a source, but here the source will be of a general multipolar order. We will try to derive a formula for the complete wavefield, including the near-field waves, by using the higher-order ray approximations. We will confine oursleves to P waves. The theory for multipolar S waves can be developed in an analogous way. First of all we will study an axially symmetric P-wave radiation, which is simpler and more comprehensible. After that, we will briefly repeat our derivation for a general case.

4.1 Axially symmetric radiation pattern

Let us assume a spherical far-field P wave in the following form:

$$\mathbf{u}^{P(0)}(\mathbf{x},t)=\mathbf{u}^{P(0)\|}(\mathbf{x},t)=\frac{R^{P(0)\|}(\vartheta)}{r}f^{(0)}\left(t-\frac{r}{\alpha}\right)\mathbf{g}^P,$$
$$\mathbf{u}^{P(0)\perp}(\mathbf{x},t)=0, \quad (30)$$

where $R^{P(0)\|}$ is the zeroth-order radiation function. In this section, we will assume the axially symmetric case: $R^{P(0)\|}=R^{P(0)\|}(\vartheta)$, where ϑ denotes the angle between the position vector of an observer and the axis of symmetry. Without loss of generality, we will identify the symmetry axis with the x_3-axis. From axial symmetry, an additional component of each higher-order term can be assumed to be polarized in the $\mathbf{g}^{SV}$ direction only. Thus we can write

$$\mathbf{u}^{P(K)}(\mathbf{x},t)=\mathbf{u}^{P(K)\|}(\mathbf{x},t)+\mathbf{u}^{P(K)\perp}(\mathbf{x},t),$$
$$\mathbf{u}^{P(K)\|}=\frac{\alpha^K R^{P(K)\|}(\vartheta)}{r^{K+1}}f^{(K)}\left(t-\frac{r}{\alpha}\right)\mathbf{g}^P,$$
$$\mathbf{u}^{P(K)\perp}=\frac{\alpha^K R^{P(K)\perp}(\vartheta)}{r^{K+1}}f^{(K)}\left(t-\frac{r}{\alpha}\right)\mathbf{g}^{SV}, \quad (31)$$

where $R^{P(K)\|}$ and $R^{P(K)\perp}$ will be called the principal and additional P-wave radiation functions of the Kth-order. Note that the definition (15) of radiation patterns $\mathcal{R}^{P(K)}$ differs from the definition of $R^{P(K)}$ by a scaling factor α^K, explicitly written in (31), where α is the P-wave velocity. We introduced this definition just for the reason of simplification of the next formulae. Using eqs (7a) and (16a), we obtain for the first-, second- and third-order radiation functions the following:

$$R^{P(1)\perp}=-\frac{dR^{P(0)\|}}{d\vartheta},$$
$$R^{P(1)\|}=-\frac{1}{2}\left[\frac{d^2R^{P(0)\|}}{d\vartheta^2}+\frac{\cos\vartheta}{\sin\vartheta}\frac{dR^{P(0)\|}}{d\vartheta}\right]+R^{P(0)\|},$$
$$R^{P(2)\perp}=-\frac{dR^{P(1)\|}}{d\vartheta}+\frac{dR^{P(0)\|}}{d\vartheta},$$
$$R^{P(2)\|}=-\frac{1}{4}\left[\frac{d^2R^{P(1)\|}}{d\vartheta^2}+\frac{\cos\vartheta}{\sin\vartheta}\frac{dR^{P(1)\|}}{d\vartheta}\right]+R^{P(1)\|}-R^{P(0)\|},$$
$$R^{P(3)\perp}=-\frac{dR^{P(2)\|}}{d\vartheta}+2\frac{dR^{P(1)\|}}{d\vartheta}-2\frac{dR^{P(0)\|}}{d\vartheta}, \quad (32)$$
$$R^{P(3)\|}=-\frac{1}{6}\left[\frac{d^2R^{P(2)\|}}{d\vartheta^2}+\frac{\cos\vartheta}{\sin\vartheta}\frac{dR^{P(2)\|}}{d\vartheta}\right]$$
$$+\frac{2}{3}R^{P(2)\|}-2R^{P(1)\|}+2R^{P(0)\|}.$$

We obtain the higher-order radiation functions similarly; they can be expressed as

$$R^{P(K)\perp}=-\frac{dR^{P(K-1)\|}}{d\vartheta}-(K-1)R^{P(K-1)\perp}$$
$$=\sum_{i=0}^{K-1}(-1)^{K+i}\frac{(K-1)!}{i!}\frac{dR^{P(i)\|}}{d\vartheta},$$
$$R^{P(K)\|}=-\frac{1}{K}\left\{\left(\frac{K(K-1)}{2}-1\right)R^{P(K-1)\|}+\frac{1}{2}\langle R^{P(K-1)\|}\rangle\right. \quad (33)$$
$$\left.+\sum_{i=0}^{K-2}(-1)^i\frac{(K-2)!}{(K-2-i)!}\langle R^{P(K-2-i)\|}\rangle\right\},$$

where the brackets $\langle\,\rangle$ denote the differential operator defined as follows:

$$\langle R\rangle=\frac{d^2R}{d\vartheta^2}+\frac{\cos\vartheta}{\sin\vartheta}\frac{dR}{d\vartheta}.$$

In contrast to (7a) and (16a), we managed to express additional as well as principal components of each higher-order term by differentiation of only lower *principal components*, as shown in (33). However, the recursive multiple differentiation in (33) still complicates the calculation of

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the higher-order ray approximations considerably. In the following, we will try to simplify formulae (33) by avoiding differentiation completely.

By expanding the radiation functions in a series of Legendre polynomials $P_n(\cos\vartheta)$ and the associated Legendre polynomials of the first order, $P_n^1(\cos\vartheta)$,

$$R^{P(K)\|}(\vartheta) = \sum_{n=0}^{\infty} r_n^{P(K)\|} P_n(\cos\vartheta),$$
$$R^{P(K)\perp}(\vartheta) = \sum_{n=0}^{\infty} r_n^{P(K)\perp} P_n^1(\cos), \tag{34}$$

we obtain

$$\langle R^{P(K)\|}\rangle = -\sum_{n=0}^{\infty} n(n+1) r_n^{P(K)\|} P_n(\cos\vartheta),$$
$$\frac{dR^{P(K)\|}}{d\vartheta} = \sum_{n=0}^{\infty} r_n^{P(K)\|} P_n^1(\cos\vartheta), \tag{35}$$

where we used the following identities for the Legendre polynomials:

$$\frac{1}{\sin\vartheta}\frac{d}{d\vartheta}\left(\sin\vartheta\frac{dP_n(\cos\vartheta)}{d\vartheta}\right) + n(n+1)P_n(\cos\vartheta) = 0,$$
$$P_n^1(\cos\vartheta) = \frac{dP_n(\cos\vartheta)}{d\vartheta}. \tag{36}$$

In (35) and (36) we adopted Hobson's definition of the associated Legendre polynomials, which yields $P_1^1(\cos\vartheta) = -\sin\vartheta$, but not another definition by Ferrer: $P_1^1(\cos\vartheta) = \sin\vartheta$.

Inserting (34) and (35) into (33), we obtain algebraic formulae for the coefficients of the Legendre expansion for the higher-order radiation functions expressed only by the zeroth-order coefficients:

$$r_n^{P(1)\|} = \tfrac{1}{2}(n^2+n+2)r_n^{P(0)}, \qquad r_n^{P(1)\perp} = -r_n^{P(0)},$$
$$r_n^{P(2)\|} = \tfrac{1}{8}(n^2+n+6)(n+1)nr_n^{P(0)},$$
$$r_n^{P(2)\perp} = -\tfrac{1}{2}(n+1)nr_n^{P(0)}, \tag{37}$$
$$r_n^{P(3)\|} = \tfrac{1}{48}(n^2+n+12)(n+2)(n+1)n(n-1)r_n^{P(0)},$$
$$r_n^{P(3)\perp} = -\tfrac{1}{8}(n+2)(n+1)n(n-1)r_n^{P(0)},$$

or generally for the Kth-order ray approximation, for $K \le n$

$$r_n^{P(K+1)\|} = \frac{1}{2^{K+1}(K+1)!}(n^2+n+(K+1)(K+2)) \times \frac{(n+K)!}{(n-K)!} r_n^{P(0)}, \tag{38}$$

$$r_n^{P(K+1)\perp} = -\frac{1}{2^K K!}\frac{(n+K)!}{(n-K)!} r_n^{P(0)},$$

and for $K > n$

$$r_n^{P(K+1)\|} = r_n^{P(K+1)} = 0.$$

$r_n^{P(0)}$ denotes the coefficient of the principal zeroth-order radiation function. Since the additional component is zero in this case, we omit a superscript denoting the principal component explicitly.

Formulae (38) represent simple algebraic forms for all the higher-order ray approximations. These approximations are expressed in terms of expansion coefficients of the zeroth-order ray approximation into a series of Legendre polynomials. It follows that from a known far-field radiation, we can easily calculate all higher-order ray approximations, physically representing the near-field waves. The degree of a multipolar source is simply related to the number of non-zero higher-order ray approximations. If the source is isotropic, only the first-order ray approximation is non-zero; if the source is a single force, the first- and the second-order approximations are non-zero. In general, we have the relation

$$K = N + 1,$$

where N denotes the order of a multipolar source and K is the order of the highest non-zero ray approximation. This fact also implies from (38) that, if the Legendre expansion of the far-field radiation has only one non-zero coefficient $r_N^{(0)}$, the expansion of the respective Kth-order near-field wave will also consist of only one non-zero coefficient $r_N^{(K)}$.

4.2 General radiation pattern

Next, we assume a general directional dependence of the zeroth-order radiation function

$$R^{P(0)\|} = R^{P(0)\|}(\vartheta, \varphi).$$

In this case, the additional component of higher-order radiation functions will be polarized in a general direction perpendicular to $\mathbf{g}^P$, thus having a component in the direction of SV- as well as SH-wave polarization vectors:

$$\mathbf{u}^{P(K)\|}(\mathbf{x}, t) = \frac{\alpha^K R^{P(K)\|}(\vartheta, \varphi)\mathbf{g}^P}{r^{K+1}} f^{(K)}\left(t - \frac{r}{\alpha}\right),$$
$$\mathbf{u}^{P(K)\perp}(\mathbf{x}, t) = \frac{\alpha^K (R^{P(K)\perp SV}\mathbf{g}^{SV} + R^{P(K)\perp SH}\mathbf{g}^{SH})}{r^{K+1}} f^{(K)}\left(t - \frac{r}{\alpha}\right). \tag{39}$$

Higher-order ray approximations are calculated in a similar way to in the previous section. Therefore, we will not give the details, but only the final formulae.

Inserting (39) into (7a) and (16a), we obtain for the higher-order radiation functions

$$R^{P(K)\perp SV} = -\frac{\partial R^{P(K-1)\|}}{\partial\vartheta} - (K-1)R^{P(K-1)\perp SV}$$
$$= \sum_{i=0}^{K-1}(-1)^{K+i}\frac{(K-1)!}{i!}\frac{\partial R^{P(i)\|}}{\partial\vartheta},$$
$$R^{P(K)\perp SH} = -\frac{1}{\sin\vartheta}\left(\frac{\partial R^{P(K-1)\|}}{\partial\varphi} + (K-1)R^{P(K-1)\perp SH}\right)$$
$$= \frac{1}{\sin\vartheta}\sum_{i=0}^{K-1}(-1)^{K+i}\frac{(K-1)!}{i!}\frac{\partial R^{P(i)\|}}{\partial\varphi}, \tag{40}$$
$$R^{P(K)\|} = -\frac{1}{K}\left\{\left(\frac{K(K-1)}{2} - 1\right)R^{P(K-1)\|} + \frac{1}{2}\langle R^{P(K-1)\|}\rangle + \sum_{i=0}^{K-2}(-1)^i\frac{(K-2)!}{(K-2-i)!}\langle R^{P(K-2-i)\|}\rangle\right\},$$

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where the brackets $\langle\rangle$ denote the Beltrami operator (see Ben-Menahem & Singh 1981, eq. I.5) defined as follows:

$$\langle R\rangle = \frac{\partial^2 R}{\partial\vartheta^2} + \frac{\cos\vartheta}{\sin\vartheta}\frac{\partial R}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2 R}{\partial\varphi^2}. \tag{41}$$

Expanding the radiation functions into the *vector spherical harmonics* (Aki & Richards 1890, eq. 8.13; Ben-Menahem & Singh 1981, eq. 2.71):

$$\begin{aligned}\mathbf{R}_n^m(\vartheta,\varphi) &= Y_n^m(\vartheta,\varphi)\mathbf{g}^P,\\ \mathbf{S}_n^m(\vartheta,\varphi) &= \frac{1}{\sqrt{n(n+1)}}\left(\frac{\partial Y_n^m}{\partial\vartheta}\mathbf{g}^{SV} + \frac{1}{\sin\vartheta}\frac{\partial Y_n^m}{\partial\varphi}\mathbf{g}^{SH}\right),\end{aligned} \tag{42}$$

we obtain

$$\begin{aligned}R^{P(K)\|}\mathbf{g}^P &= \sum_{n,m=0}^{\infty} r_{nm}^{P(K)\|}\mathbf{R}_n^m(\vartheta,\varphi),\\ R^{P(K)\perp SV}\mathbf{g}^{SV} + R^{P(K)\perp SH}\mathbf{g}^{SH} &= \sum_{n,m=0}^{\infty} r_{nm}^{P(K)\perp}\mathbf{S}_n^m(\vartheta,\varphi).\end{aligned} \tag{43}$$

Taking the following identity into account (Ben-Menahem & Singh 1981, eq. I.8),

$$\begin{aligned}&\frac{1}{\sin\vartheta}\frac{\partial}{\partial\vartheta}\left(\sin\vartheta\frac{\partial Y_n^m(\vartheta,\varphi)}{\partial\vartheta}\right) + \frac{1}{\sin^2\vartheta}\frac{\partial^2 Y_n^m(\vartheta,\varphi)}{\partial\varphi^2}\\ &\quad + n(n+1)Y_n^m(\vartheta,\varphi) = 0,\end{aligned} \tag{44}$$

we obtain

$$\langle R^{P(K)\|}\rangle = -\sum_{n,m=0}^{\infty} n(n+1) r_{nm}^{P(K)\|} Y_n^m(\vartheta,\varphi). \tag{45}$$

Finally, the coefficients of the spherical harmonics expansion of the Kth-order radiation functions are expressed, for $K \le n$, by

$$\begin{aligned}r_{nm}^{P(K+1)\|} &= \frac{1}{2^{K+1}(K+1)!}(n^2+n+(K+1)(K+2))\\ &\quad\times\frac{(n+K)!}{(n-K)!}r_{nm}^{P(0)},\\ r_{nm}^{P(K+1)\perp} &= -\frac{\sqrt{n(n+1)}}{2^K K!}\frac{(n+K)!}{(n-K)!}r_{nm}^{P(0)},\end{aligned} \tag{46}$$

and, for $K > n$, by

$$r_{nm}^{P(K+1)\|} = r_{nm}^{P(K+1)\perp} = 0.$$

We again omit a superscript denoting the principal component explicitly for the zeroth-order coefficient.

5 EXAMPLES

5.1 Isotropic source

Let us assume an isotropic source at the origin of coordinates with an impulse-like source-time function. The generated wavefield consists only of the spherically symmetric P wave, which can be expressed in the far field as

$$\mathbf{u}^{P(0)}(\mathbf{x},t) = \frac{R^{P(0)\|}\mathbf{g}^P}{r}\dot{\delta}\left(t-\frac{r}{\alpha}\right), \quad \text{where } R^{P(0)\|} = \frac{1}{4\pi\rho\alpha^3}. \tag{47}$$

Expanding $R^{P(0)\|}$ into a series of Legendre polynomials according to (34), we find that the only non-zero term of the expansion is $r_0^{P(0)}$:

$$r_0^{P(0)\|} = \frac{1}{4\pi\rho\alpha^3}. \tag{48}$$

According to (38), the non-zero coefficients of the principal and additional components of higher-order approximations are

$$r_0^{P(1)\|} = r_0^{P(0)} \quad\text{and}\quad r_0^{P(1)\perp} = -r_0^{P(0)}. \tag{49}$$

Since $P_0^1(\cos\vartheta)$ is identically equal to zero, the additional component $\mathbf{u}^{P(1)\perp}(\mathbf{x},t)$ vanishes, and the complete wavefield is

$$\begin{aligned}\mathbf{u}^P(\mathbf{x},t) &= \frac{R^{P(0)\|}\mathbf{g}^P}{r}\dot{\delta}\left(t-\frac{r}{\alpha}\right) + \frac{\alpha R^{P(1)\|}\mathbf{g}^P}{r^2}\delta\left(t-\frac{r}{\alpha}\right)\\ &= \frac{\mathbf{g}^P}{4\pi\rho\alpha^2}\left[\frac{1}{\alpha r}\dot{\delta}\left(t-\frac{r}{\alpha}\right) + \frac{1}{r^2}\delta\left(t-\frac{r}{\alpha}\right)\right].\end{aligned} \tag{50}$$

Formula (50) is well known and it can be found in, for example, Ben-Menahem & Singh (1981, eq. 4.208). The wavefield for a general form of the source-time function can be obtained from (50) by applying the time convolution operator.

5.2 Double-couple source

Next, we will examine a wavefield generated by the double-couple source at the origin of coordinates with the moment tensor $M_{ij}(t)$:

$$M_{ij}(t) = (\delta_{1i}\delta_{3j} + \delta_{1j}\delta_{3i})M(t) = (\delta_{1i}\delta_{3j} + \delta_{1j}\delta_{3i})\delta(t). \tag{51}$$

As for the isotropic source, we assume in (51) that the source-time function $M(t)$ is the Dirac delta function. Obviously, any other form of $M(t)$ (for example the step function frequently used in source modelling studies) can easily be incorporated in all the following formulae by the time convolution. The far-field P-wave displacement $\mathbf{u}^{P(0)}(\mathbf{x},t)$ is expressed (Aki & Richards 1980, eq. 4.29) by

$$\mathbf{u}^{P(0)}(\mathbf{x},t) = \frac{g_1^P g_3^P}{2\pi\rho\alpha^3}\frac{\mathbf{g}^P}{r}\dot{\delta}\left(t-\frac{r}{\alpha}\right), \tag{52}$$

and thus the zeroth-order radiation function is

$$R^{P(0)\|} = \frac{g_1^P g_3^P}{2\pi\rho\alpha^3} = \frac{1}{2\pi\rho\alpha^3}\sin\vartheta\cos\vartheta\cos\varphi. \tag{53}$$

Expanding $R^{P(0)\|}\mathbf{g}^P$ into the vector spherical harmonics (43), we find that the only non-zero term of the expansion is $r_{21}^{P(0)}$:

$$r_{21}^{P(0)\|} = \frac{1}{2\pi\rho\alpha^3}\sqrt{\frac{8\pi}{15}}. \tag{54}$$

According to (46), we have, for the coefficients of the principal and additional components of higher-order approximations,

$$\begin{aligned}&r_{21}^{P(1)\|} = 4r_{21}^{P(0)}, \quad r_{21}^{P(2)\|} = 9r_{21}^{P(0)}, \quad r_{21}^{P(3)\|} = 9r_{21}^{P(0)},\\ &r_{21}^{P(0)\perp} = -\sqrt{6}\,r_{21}^{P(0)}, \quad r_{21}^{P(2)\perp} = -3\sqrt{6}\,r_{21}^{P(0)},\\ &r_{21}^{P(3)\perp} = -3\sqrt{6}\,r_{21}^{P(0)},\end{aligned} \tag{55}$$

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all other coefficients of the expansion being zero. Taking the following equations into account:

$$\mathbf{R}_2^1(\vartheta, \varphi) = \sqrt{\frac{15}{8\pi}} g_1^P g_3^P \mathbf{g}^P,$$

$$\mathbf{S}_2^1(\vartheta, \varphi) = \frac{1}{\sqrt{6}} \left(\frac{\partial Y_2^1}{\partial \vartheta} \mathbf{g}^{SV} + \frac{1}{\sin \vartheta} \frac{\partial Y_2^1}{\partial \varphi} \mathbf{g}^{SH} \right) \qquad (56)$$

$$= \frac{1}{4} \sqrt{\frac{5}{\pi}} \{ -2 g_1^P g_3^P \mathbf{g}^P + g_3^P \mathbf{e}^1 + g_1^P \mathbf{e}^3 \},$$

where $\mathbf{e}^1 = (1, 0, 0)^T$ and $\mathbf{e}^3 = (0, 0, 1)^T$, we obtain, for higher-order radiation functions,

$$R_i^{P(1)} = \frac{1}{2\pi\rho\alpha^3} (6 g_1^P g_3^P g_i^P - g_3^P \delta_{1i} - g_1^P \delta_{3i}),$$

$$R_i^{P(2)} = \frac{1}{2\pi\rho\alpha^3} (15 g_1^P g_3^P g_i^P - 3 g_3^P \delta_{1i} - 3 g_1^P \delta_{3i}), \qquad (57)$$

$$R_i^{P(3)} = \frac{1}{2\pi\rho\alpha^3} (15 g_1^P g_3^P g_i^P - 3 g_3^P \delta_{1i} - 3 g_1^P \delta_{3i}).$$

For the complete P wave, we can finally write

$$u_i^P(\mathbf{x}, t) = \frac{1}{2\pi\rho} \left\{ g_1^P g_3^P g_i^P \frac{1}{\alpha^3 r} \dot{\delta}\left(t - \frac{r}{\alpha}\right) \right.$$

$$+ (6 g_1^P g_3^P g_i^P - g_3^P \delta_{1i} - g_1^P \delta_{3i}) \frac{1}{\alpha^2 r^2} \delta\left(t - \frac{r}{\alpha}\right)$$

$$\left. + (15 g_1^P g_3^P g_i^P - 3 g_3^P \delta_{1i} - 3 g_1^P \delta_{3i}) \frac{t}{r^4} H\left(t - \frac{r}{\alpha}\right) \right\}. \qquad (58)$$

Formula (58) coincides with the exactly calculated P wave generated by the double-couple source (Aki & Richards 1980, eq. 4.30). The formula for the S wave can be obtained in an analogous way.

6 CONCLUSION

In our paper, we show that an exact formula for the Green's function for a homogeneous isotropic and unbounded medium can be obtained by the ray method. The ray series of the Green's function can be calculated by using the basic equations of ray theory and it consists only of three non-zero terms. The zeroth-order term corresponds to the far-field wave. The higher-order ray approximations correspond to the near-field waves, which are neglected by the geometrical ray method.

By analytical calculations of the higher-order ray approximations of the wavefields generated by multipolar point sources, we found a simple algebraic relation between the coefficients of the spherical harmonics expansion of the far-field radiation function and the radiation of the near-field waves. We show that a multipolar source of the Nth order generates a wavefield *exactly* expressed by the ray series with $N+2$ non-zero terms (including the zeroth-order term): all higher-order ray approximations equal zero. We found that the form of the principal component of the radiation function for the near-field waves follows the form of the far-field radiation function: meaning that it consists of the same spherical harmonics functions. The higher-order ray approximations are expressed by the higher-order spherical harmonics, the low-order coefficients of the expansion being zero.

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