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Real ray tracing in anisotropic viscoelastic media

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SUMMARY

Ray tracing equations applicable to smoothly inhomogeneous anisotropic viscoelastic media are derived. The equations produce real rays, in contrast to previous ray-theoretical approaches, which deal with complex rays. The real rays are defined as the solutions of the Hamilton equations, with the Hamiltonian modified for viscoelastic media, and physically correspond to trajectories of high-frequency waves characterized by a real stationary phase. As a consequence, the complex eikonal equation is satisfied only approximately. The ray tracing equations are valid for weakly and moderately attenuating media. The rays are frequency-dependent and must be calculated for each frequency, separately.

Solving the ray tracing equations in viscoelastic anisotropy is more time consuming than in elastic anisotropy. The main difficulty is with determining the stationary slowness vector, which is generally complex-valued and inhomogeneous and must be computed at each time step of the ray tracing procedure. In viscoelastic isotropy, the ray tracing equations considerably simplify, because the stationary slowness vector is homogeneous. The computational time for tracing rays in isotropic elastic and viscoelastic media is the same. Using numerical examples, it is shown that ray fields in weakly attenuating media (Q higher than about 30) are almost indistinguishable from those in elastic media. For moderately attenuating anisotropic media (Q between 5–20), the differences in ray fields can be visible and significant.

Key words: Elasticity and anelasticity; Body waves; Seismic anisotropy; Seismic attenuation; Wave propagation.

1 INTRODUCTION

The attenuation and dispersion of waves propagating in real rock structures is frequently modelled, using a linearized viscoelastic medium. Properties of such a medium are described by frequency-dependent elasticity and viscosity parameters, which can formally be integrated into complex-valued viscoelastic parameters. In the frequency domain, the wave propagation problems are solved in a similar way as in elasticity, except that some of wave attributes in viscoelasticity become complex valued (Auld 1973; Carcione 2007). In homogeneous viscoelastic media, this approach has been applied to studying properties of plane waves (Caviglia & Morro 1992; Shuvalov & Scott 1999; Shuvalov 2001; Červený & Pšenčík 2005; Zhu & Tsvankin 2006, 2007) and to calculating the exact and asymptotic Green's functions and other asymptotic wave quantities (Carcione 1994; Vavryčuk 2007a,b). In inhomogeneous viscoelastic media, the complete wavefields are usually computed using time-demanding numerical methods that directly solve the wave equation (Carcione 1990; Saenger & Bohlen 2004; Moczo *et al.* 2004, 2007).

An alternative to the direct numerical methods is to apply ray theory (Červený 2001) and to compute wavefields propagating in 3-D inhomogeneous viscoelastic media using high-frequency approximation. In this case, the most straightforward way is to construct rays and other ray quantities in the elastic background medium and to incorporate the effects of attenuation as perturbations (Gajewski & Pšenčík 1992; Vavryčuk 2008). This procedure is simple and effective but applicable only to weakly attenuating media. Another option is to apply the complex ray theory, which deals with rays as trajectories in complex space (Hearn & Krebes 1990a,b; Le *et al.* 1994; Thomson 1997; Chapman *et al.* 1999; Kravtsov *et al.* 1999; Hanyga & Seredyňska 2000; Kravtsov 2005). At the first sight, this theory seems to be mathematically elegant and straightforward derivable, but in fact, it proves awkward and far more complicated than real ray theory. So far, it has not been fully developed and struggles with essential difficulties, which prevent from using it in realistic seismic applications.

In this paper, I develop novel ray tracing equations, applicable to smoothly inhomogeneous anisotropic viscoelastic media. The ray tracing equations are derived from the Hamilton equations by using the Hamiltonian modified for viscoelastic media. The equations produce real-valued rays in contrary to the approach dealing with complex rays. The rays are defined as trajectories of energy transported by high-frequency waves with a real stationary phase. The ray tracing equations are valid for weakly and moderately attenuating media. Using numerical examples, the applicability of the ray tracing equations is tested and the differences

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between ray fields and traveltimes in elastic and viscoelastic media are discussed.

2 DEFINITION OF THE VISCOELASTIC MEDIUM

2.1 Notations

In formulae, the real and imaginary parts of complex-valued quantities are denoted by superscripts R and I, respectively. A complex-conjugate quantity is denoted by an asterisk. The direction of a complex-valued vector $\mathbf{v}$ is calculated as $\mathbf{v}/\sqrt{\mathbf{v}^T\mathbf{v}}$, where superscript T means the transposition (the normalization condition $\mathbf{v}/\sqrt{\mathbf{v}\cdot\mathbf{v}^*} = \sqrt{v_1 v_1^* + v_2 v_2^* + v_3 v_3^*}$ is not used). The magnitude of complex-valued vector $\mathbf{v}$ is complex valued and is calculated as $v = \sqrt{\mathbf{v}^T\mathbf{v}} = \sqrt{v_1 v_1 + v_2 v_2 + v_3 v_3}$.

If any complex-valued vector is defined by a real-valued direction, it is called homogeneous and if defined by a complex-valued direction, it is called inhomogeneous. The directions of the real and imaginary parts of a complex-valued vector are parallel for a homogeneous vector but non-parallel for an inhomogeneous vector. The terms 'homogeneous' and 'inhomogeneous' are also used in the text to describe properties of the medium: elastic/viscoelastic parameters are constant in homogeneous media, but spatially-dependent in inhomogeneous media. From the context, it is clear in which sense the terms 'homogeneous/inhomogeneous' are used.

Besides the standard four-index notation for viscoelastic parameters a_{ijkl} and quality parameters q_{ijkl}, also the two-index Voigt notation A_{MN} and Q_{MN} is alternatively used. The Voigt notation reduces pairs of indices i, j or k, l into a single index M or N using the following rules

$$11 \to 1,\ 22 \to 2,\ 33 \to 3,\ 23 \to 4,\ 13 \to 5 \text{ and } 12 \to 6. \quad (1)$$

Quantities in the frequency domain are calculated using the Fourier transform defined as follows

$$f(\omega) = F[f(t)] = \int_{-\infty}^{\infty} f(t)\exp(\mathrm{i}\omega t)\,\mathrm{d}t. \quad (2)$$

In formulae, the Einstein summation convention is used for repeated subscripts.

2.2 Viscoelastic parameters

A viscoelastic medium is defined by density-normalized viscoelastic parameters a_{ijkl}, which are, in general, complex-valued, frequency-dependent and vary with position vector $\mathbf{x}$. The real and imaginary parts of a_{ijkl},

$$a_{ijkl}(\mathbf{x},\omega) = a^{\mathrm{R}}_{ijkl} + \mathrm{i}\,a^{\mathrm{I}}_{ijkl}, \quad (3)$$

define elastic and viscous properties of the medium. We assume that viscoelastic parameters a_{ijkl} satisfy the symmetry relations

$$a_{ijkl} = a_{jikl} = a_{ijlk} = a_{klij}. \quad (4)$$

The ratio between the real and imaginary parts of a_{ijkl} is called the matrix of quality factor parameters,

$$q_{ijkl}(\mathbf{x},\omega) = -\frac{a^{\mathrm{R}}_{ijkl}}{a^{\mathrm{I}}_{ijkl}} \text{ (no summation over repeated indices)}, \quad (5)$$

and quantifies how attenuative the medium is. Obviously, q_{ijkl} is not a tensor quantity. The sign in eq. (5) depends on the definition of the Fourier transform (2), used for calculating the viscoelastic parameters in the frequency domain. When using the Fourier transform with the exponential term $\exp(-\mathrm{i}\omega t)$, the minus sign in (5) must be omitted.

2.3 Equation of motion and the eikonal equation

The equation of motion for a smoothly inhomogeneous anisotropic viscoelastic medium, when no sources are considered, reads (Červený 2001, eq. 2.1.27)

$$\rho\,\omega^2 u_i + \left(\rho\, a_{ijkl} u_{k,l}\right)_{,j} = 0,\ \ i = 1, 2, 3, \quad (6)$$

where $\mathbf{u} = \mathbf{u}(\mathbf{x},\omega)$ is the displacement, $\rho = \rho(\mathbf{x})$ is the density of the medium, $a_{ijkl} = a_{ijkl}(\mathbf{x},\omega)$ are the density-normalized viscoelastic parameters and ω is the circular frequency. We assume that viscoelastic parameters a_{ijkl} and density ρ and their derivatives are continuous functions of coordinates. Frequency ω, density ρ and position vector $\mathbf{x}$ are real valued, viscoelastic parameters a_{ijkl} and displacement $\mathbf{u}$ are complex valued.

The displacement $\mathbf{u} = \mathbf{u}(\mathbf{x},\omega)$ is assumed to describe a high-frequency harmonic signal expressed as

$$u_i(\mathbf{x},\omega) = U_i(\mathbf{x})\exp[\mathrm{i}\omega\tau(\mathbf{x})], \quad (7)$$

where $\mathbf{U} = \mathbf{U}(\mathbf{x})$ is the complex-valued ray amplitude and $\tau = \tau(\mathbf{x})$ is the complex-valued traveltime. Inserting eq. (7) into the equation of motion (6), we obtain the eikonal equation in the form

$$G(\mathbf{x},\mathbf{p}) = a_{ijkl} p_i p_l g_j g_k = 1, \quad (8)$$

where G is the eigenvalue and $\mathbf{g}$ is the eigenvector of the Christoffel tensor of the studied wave (P, $S1$ or $S2$),

$$\Gamma_{jk}(\mathbf{x},\mathbf{p}) = a_{ijkl} p_i p_l, \quad (9)$$

and vector $\mathbf{p}$ is the complex-valued slowness vector defined as

$$p_i = \frac{\partial\tau}{\partial x_i}. \quad (10)$$

Eigenvector $\mathbf{g}$ is also called the polarization vector.

3 RAY TRACING IN ANISOTROPIC MEDIA: ELASTIC CASE

3.1 Generalized coordinates and the Hamilton equations

If the medium is elastic, the stiffness parameters a_{ijkl} are real valued and frequency-independent. Consequently, the slowness vector $\mathbf{p}$, polarization vector $\mathbf{g}$ and the traveltime τ are also real valued. The eikonal equation can be rewritten in the following general form (Červený 2001, eq. 3.6.3)

$$H(\mathbf{x},\mathbf{p}) = \frac{1}{2}[G(\mathbf{x},\mathbf{p}) - 1] = 0, \quad (11)$$

where $H = H(\mathbf{x},\mathbf{p})$ is called the Hamiltonian and vectors $\mathbf{x}$ and $\mathbf{p}$ are called the generalized coordinates. The eikonal equation in the Hamiltonian form (11) represents a non-linear partial differential equation for the traveltime $\tau = \tau(\mathbf{x})$. The ray tracing equations can be obtained from (11) using a method of characteristics (Courant & Hilbert 1962). The method of characteristics transforms the partial differential eq. (11) into a system of ordinary differential equations. The characteristics are expressed in the Hamiltonian canonical form as follows (Červený 2002, eq. 5)

$$\frac{\mathrm{d}x_i}{\mathrm{d}\sigma} = \frac{\partial H}{\partial p_i},\ \frac{\mathrm{d}p_i}{\mathrm{d}\sigma} = -\frac{\partial H}{\partial x_i},\ \frac{\mathrm{d}\tau}{\mathrm{d}\sigma} = p_i\frac{\partial H}{\partial p_i}, \quad (12)$$

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where σ is a parameter along a ray. For $\sigma = \tau$, eq. (12) simplifies, taking the following form (see Červený 2001, eq. 3.6.4)

$$\frac{\mathrm{d}x_i}{\mathrm{d}\tau} = \frac{1}{2}\frac{\partial G}{\partial p_i}, \quad \frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{1}{2}\frac{\partial G}{\partial x_i}, \tag{13}$$

where G is given by eq. (8).

3.2 Ray tracing equations

Taking into account that stiffness parameters a_{ijkl} depend on $\mathbf{x}$, $a_{ijkl} = a_{ijkl}(\mathbf{x})$ and the unit polarization vector $\mathbf{g}$ depends on $\mathbf{p}$, $\mathbf{g} = \mathbf{g}(\mathbf{p})$, we obtain the ray tracing equations in the following form (see Červený 2001, eq. 3.6.10; Vavryčuk 2001, eq. 7)

$$\frac{\mathrm{d}x_i}{\mathrm{d}\tau} = a_{ijkl}p_l g_j g_k, \tag{14}$$

$$\frac{\mathrm{d}p_i}{\mathrm{d}\tau} = -\frac{1}{2}\frac{\partial a_{jkln}}{\partial x_i} p_k p_n g_j g_l. \tag{15}$$

Eq. (14) describes the velocity of the signal propagating along a ray. The velocity is denoted as $\mathbf{v}$ and called the group velocity vector (see Červený 2001) or energy velocity vector (see Carcione 2007). Eq. (15) describes the time variation of the slowness vector along a ray.

Multiplying eq. (14) by the slowness vector $\mathbf{p}$ and taking into account eq. (8), we obtain the following well-known identity

$$v_i p_i = 1. \tag{16}$$

Eqs (14) and (15) represent a system of six ordinary differential equations of the first order, which can be solved in a standard way (see Press *et al.* 1992). Some problems can arise near singularities (acoustic axes) in anisotropic media (Vavryčuk 2005), where rays must be traced with care. Otherwise, false and unphysical bending of rays can occur when a ray passes the singularity (Vavryčuk 2001, 2003).

4 RAY TRACING IN ANISOTROPIC MEDIA: VISCOELASTIC CASE

4.1 Generalized coordinates and the Hamilton equations

In viscoelastic media, the problem of ray tracing is more involved. First, parameters a_{ijkl} are frequency-dependent, hence tracing rays must be done for each frequency separately. Second, since parameters a_{ijkl} are complex valued, slowness vector $\mathbf{p}$, polarization vector $\mathbf{g}$ and traveltime τ become complex valued. If we restrict the ray to be a trajectory in the real space, the vector $\mathbf{v}$ defined as

$$v_i = \frac{\mathrm{d}x_i}{\mathrm{d}\tau} = a_{ijkl}p_l g_j g_k \tag{17}$$

is a complex-valued homogeneous vector (see Appendix), called the complex energy velocity vector (see Vavryčuk 2007a,b). Its magnitude $v = \sqrt{\mathbf{v}^T\mathbf{v}}$, is called the complex energy velocity. Note that 'complex energy velocity' is different from the 'energy velocity' used by Carcione (2006, 2007), where it means a real-valued time-averaged quantity.

Since the complex energy velocity vector $\mathbf{v}$ is homogeneous, the real and imaginary parts of $\mathbf{v}$ are parallel. This condition restricts possible values of the slowness vector $\mathbf{p}$ and indicates that the real and imaginary parts of $\mathbf{p}$ are not independent in the ray tracing equations. The slowness vector $\mathbf{p}$, which predicts a homogeneous complex energy velocity vector $\mathbf{v}$, is called the 'stationary' slowness vector (see Vavryčuk 2007a,b). The stationary slowness vector is, in general, inhomogeneous. Since the real and imaginary parts of $\mathbf{p}$ are not independent, vector $\mathbf{p}$ cannot be used as the generalized variable in the Hamilton equations (12). Instead, we can define the real-valued vectors $\mathbf{x}$ and $\mathbf{p}^{\mathrm{R}}$,

$$p_i^{\mathrm{R}} = \frac{\partial \tau^{\mathrm{R}}}{\partial x_i}, \tag{18}$$

as the generalized coordinates being functions of the real-valued traveltime τ^{R}. An inverse quantity to $\mathbf{p}^{\mathrm{R}}$ is the ray velocity $\mathbf{V}^{\mathrm{ray}}$,

$$V_i^{\mathrm{ray}} = \frac{\mathrm{d}x_i}{\mathrm{d}\tau^{\mathrm{R}}}, \tag{19}$$

which physically means the velocity of a high-frequency signal with a real stationary phase propagating along a ray. For a discussion clarifying the differences between the 'ray velocity', 'complex energy velocity' and 'energy velocity', see Vavryčuk (2007b).

Eqs (18) and (19) imply that a similar identity to eq. (16) valid for vectors $\mathbf{p}$ and $\mathbf{v}$ can be established also for vectors $\mathbf{p}^{\mathrm{R}}$ and $\mathbf{V}^{\mathrm{ray}}$,

$$V_j^{\mathrm{ray}} p_j^{\mathrm{R}} = 1. \tag{20}$$

Considering the Hamiltonian $H = H\left(\mathbf{x}, \mathbf{p}^{\mathrm{R}}\right)$ in the same form as in eq. (11), the ray tracing equations read

$$\frac{\mathrm{d}x_i}{\mathrm{d}\tau^R} = \frac{1}{2}\frac{\partial G}{\partial p_i^{\mathrm{R}}}, \quad \frac{\mathrm{d}p_i^{\mathrm{R}}}{\mathrm{d}\tau^{\mathrm{R}}} = -\frac{1}{2}\frac{\partial G}{\partial x_i}, \tag{21}$$

where

$$G\left(\mathbf{x}, \mathbf{p}^{\mathrm{R}}\right) = a_{ijkl}p_i p_l g_j g_k = 1. \tag{22}$$

As in the elastic case, the polarization vector $\mathbf{g}$ depends on $\mathbf{p}$, $\mathbf{g} = \mathbf{g}(\mathbf{p})$, but the slowness vector $\mathbf{p}$ must further be decomposed into its real and imaginary parts,

$$p_j = p_j^{\mathrm{R}} + \mathrm{i}p_j^{\mathrm{I}}. \tag{23}$$

The imaginary part $\mathbf{p}^{\mathrm{I}}$ is not independent, but a function of a_{ijkl} and $\mathbf{p}^{\mathrm{R}}$, $\mathbf{p}^{\mathrm{I}} = \mathbf{p}^{\mathrm{I}}\left(a_{ijkl}, \mathbf{p}^{\mathrm{R}}\right)$.

4.2 The first ray tracing equation

Let us treat the first equation of (21)

$$V_i^{\mathrm{ray}} = \frac{1}{2}\frac{\partial G}{\partial p_i^{\mathrm{R}}} = \frac{1}{2}\frac{\partial G}{\partial p_k}\frac{\partial p_k}{\partial p_i^{\mathrm{R}}} = v_k\frac{\partial p_k}{\partial p_i^{\mathrm{R}}} = v_k\left(\delta_{ik} + \mathrm{i}\frac{\partial p_k^{\mathrm{I}}}{\partial p_i^{\mathrm{R}}}\right). \tag{24}$$

Decomposing the complex energy velocity $\mathbf{v}$ into its real and imaginary parts and taking into account that the ray velocity $\mathbf{V}^{\mathrm{ray}}$ is real valued, we can get from (24) the following equations,

$$v_i^{\mathrm{I}} + v_k^{\mathrm{R}}\frac{\partial p_k^{\mathrm{I}}}{\partial p_i^{\mathrm{R}}} = 0, \tag{25}$$

$$v_i^{\mathrm{R}} - v_k^{\mathrm{I}}\frac{\partial p_k^{\mathrm{I}}}{\partial p_i^{\mathrm{R}}} = V_i^{\mathrm{ray}}. \tag{26}$$

Since vector $\mathbf{v}$ is homogeneous (see Appendix), its real and imaginary parts are parallel,

$$v_i^{\mathrm{R}} = N_i v^{\mathrm{R}}, \quad v_i^{\mathrm{I}} = N_i v^{\mathrm{I}}, \tag{27}$$

where **N** is the real-valued direction vector. Inserting eq. (27) into (25) and multiplying by **N**, we obtain

$$N_i N_k \frac{\partial p_i}{\partial p_k} = -\frac{v^{\mathrm{I}}}{v^{\mathrm{R}}}. \tag{28}$$

Multiplying eq. (26) by **N** and inserting eq. (28) into (26), we finally get for $\mathbf{V}^{\mathrm{ray}}$

$$V_i^{\mathrm{ray}} = N_i \frac{v^{\mathrm{R}} v^{\mathrm{R}} + v^{\mathrm{I}} v^{\mathrm{I}}}{v^{\mathrm{R}}}. \tag{29}$$

This equation is identical with eqs (21) and (23) of Vavryčuk (2007b), derived for the ray velocity of asymptotic wavefields generated by point sources and propagating in homogeneous anisotropic viscoelastic media.

4.3 The second ray tracing equation

The second equation of (21) reads

$$\frac{\mathrm{d}p_i^{\mathrm{R}}}{\mathrm{d}\tau^{\mathrm{R}}} = -\frac{1}{2}\frac{\partial G}{\partial x_i} = -\frac{1}{2}\left[\frac{\partial G}{\partial a_{jklm}}\frac{\partial a_{jklm}}{\partial x_i} + \frac{\partial G}{\partial p_k}\frac{\partial p_k}{\partial x_i}\right]$$
$$= -\frac{1}{2}\frac{\partial a_{jklm}}{\partial x_i} p_j p_m g_k g_l - \mathrm{i} v_k \frac{\partial p_k^{\mathrm{I}}}{\partial x_i}, \tag{30}$$

where we used

$$\frac{\partial G}{\partial p_k} = 2v_k \text{ and } \frac{\partial p_k}{\partial x_i} = \mathrm{i}\frac{\partial p_k^{\mathrm{I}}}{\partial x_i}. \tag{31}$$

Since the left-hand side of (30) is real valued, we get

$$-\frac{1}{2}\left[\frac{\partial a_{jklm}}{\partial x_i} p_j p_m g_k g_l\right]^{\mathrm{I}} - v_k^{\mathrm{R}} \frac{\partial p_k^{\mathrm{I}}}{\partial x_i} = 0, \tag{32}$$

$$-\frac{1}{2}\left[\frac{\partial a_{jklm}}{\partial x_i} p_j p_m g_k g_l\right]^{\mathrm{R}} + v_k^{\mathrm{I}} \frac{\partial p_k^{\mathrm{I}}}{\partial x_i} = \frac{\mathrm{d}p_i^{\mathrm{R}}}{\mathrm{d}\tau^{\mathrm{R}}}. \tag{33}$$

From eqs (27) and (32), we can write

$$v_k^I \frac{\partial p_k^{\mathrm{I}}}{\partial x_i} = -\frac{1}{2}\frac{v^{\mathrm{I}}}{v^{\mathrm{R}}}\left[\frac{\partial a_{jklm}}{\partial x_i} p_j p_m g_k g_l\right]^{\mathrm{I}}, \tag{34}$$

and inserting it into eq. (30), we finally obtain

$$\frac{\mathrm{d}p_i^{\mathrm{R}}}{\mathrm{d}\tau^{\mathrm{R}}} = -\frac{1}{2}\left[\left(\frac{\partial a_{jklm}}{\partial x_i} p_j p_m g_k g_l\right)^{\mathrm{R}} + \frac{v^{\mathrm{I}}}{v^{\mathrm{R}}}\left(\frac{\partial a_{jklm}}{\partial x_i} p_j p_m g_k g_l\right)^{\mathrm{I}}\right]. \tag{35}$$

Obviously, if the parameters a_{ijkl} are real valued, the ray tracing eqs (29) and (35) become identical with the ray tracing equations derived for the elastic media.

5 RAY TRACING IN ISOTROPIC VISCOELASTIC MEDIA

5.1 Eikonal equation

The above derived ray tracing equations can readily be modified for isotropic media. Taking into account that the stiffness parameters c_{ijkl} are expressed in isotropic media as follows

$$c_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right), \tag{36}$$

we obtain the eigenvalue of the Christoffel tensor in the following form

$$G\left(\mathbf{x}, \mathbf{p}^{\mathrm{R}}\right) = c^2 p_i p_i = 1, \tag{37}$$

where c is the complex-valued phase velocity, being expressed either as $c = \sqrt{(\lambda + 2\mu)/\rho}$ for the P wave or as $c = \sqrt{\mu/\rho}$ for the S wave. Quantities λ and μ are the complex-valued Lamé's coefficients. Vector **p** is the complex-valued slowness vector, $p_i = \frac{\partial\tau}{\partial x_i}$, its magnitude is $p = 1/c$, vector $\mathbf{p}^{\mathrm{R}}$ is its real-valued part, $p_i^{\mathrm{R}} = \frac{\partial\tau^{\mathrm{R}}}{\partial x_i}$, and τ is the complex-valued traveltime.

5.2 Ray tracing equations

Inserting eq. (37) into eq. (17), the complex energy velocity vector **v** reads

$$v_i = c^2 p_i. \tag{38}$$

Taking into account that $p = 1/c$, we readily obtain $v = c$. Since **v** is homogeneous in (38), the slowness vector **p** must also be homogeneous. This significantly simplifies the ray tracing problem, the eqs (29) and (35) being reduced to the following form:

$$\frac{\mathrm{d}x_i}{\mathrm{d}\tau^{\mathrm{R}}} = V^2 p_i^{\mathrm{R}}, \quad \frac{\mathrm{d}p_i^{\mathrm{R}}}{\mathrm{d}\tau^{\mathrm{R}}} = -\frac{1}{V}\frac{\partial V}{\partial x_i}, \tag{39}$$

where V is the real-valued phase velocity calculated from the complex-valued phase velocity c as,

$$V = \frac{1}{\left(c^{-1}\right)^{\mathrm{R}}}, \tag{40}$$

and τ^R is the real part of the traveltime τ. Obviously, $V^{\mathrm{ray}} = V$ and $p^{\mathrm{R}} = 1/V^{\mathrm{ray}}$. The imaginary part of **p** has magnitude

$$p^{\mathrm{I}} = \left(c^{-1}\right)^{\mathrm{I}}, \tag{41}$$

and is parallel to $\mathbf{p}^{\mathrm{R}}$, $\mathbf{p}^{\mathrm{I}} \,||\, \mathbf{p}^{\mathrm{R}}$.

Note that when solving the complex eikonal equation (11) exactly without restricting the ray to be a real trajectory, the collinearity of $\mathbf{p}^{\mathrm{R}}$ and $\mathbf{p}^{\mathrm{I}}$ is lost.

6 NUMERICAL PROCEDURE

When tracing rays in anisotropic elastic media, we have to solve a system of six ordinary differential equations for real-valued vectors **x** and **p**. The right-hand sides depend on $a_{ijkl}(\mathbf{x})$ and **p**. When tracing rays in anisotropic viscoelastic media, the procedure is more involved. We have to solve a system of six ordinary differential equations for real-valued vectors **x** and $\mathbf{p}^{\mathrm{R}}$, but the right-hand sides of the equations depend not only on $a_{ijkl}(\mathbf{x}, \omega)$ and $\mathbf{p}^{\mathrm{R}}$ but also on $\mathbf{p}^{\mathrm{I}} = \mathbf{p}^{\mathrm{I}}\left(a_{ijkl}(\mathbf{x}, \omega), \mathbf{p}^{\mathrm{R}}\right)$. This means that we have to additionally calculate vector $\mathbf{p}^{\mathrm{I}}$ at each time step. This can be done by iterations, using the condition that the slowness vector **p** is stationary. For a fixed value of $\mathbf{p}^{\mathrm{R}}$, we vary $\mathbf{p}^{\mathrm{I}}$ in such a way to minimize the imaginary part of the ray direction vector **N**, $\mathbf{N} = \mathbf{v}/v$. Usually, the value of $\mathbf{p}^{\mathrm{I}}$ is much smaller than that of $\mathbf{p}^{\mathrm{R}}$, and several iterations lead to success. Nevertheless, performing the iterations at each step of ray tracing causes that calculating rays in viscoelastic media is slower than in elastic media.

Specifically, when calculating rays, we can proceed in the following way (see Fig. 1):

(1) We specify the initial conditions by setting (a) the starting point $\mathbf{x}_0$ of a ray and the initial direction of the wave normal $\mathbf{n}_0^{\mathrm{R}}$ or (b) the starting point $\mathbf{x}_0$ and the initial direction of a ray $\mathbf{N}_0$.

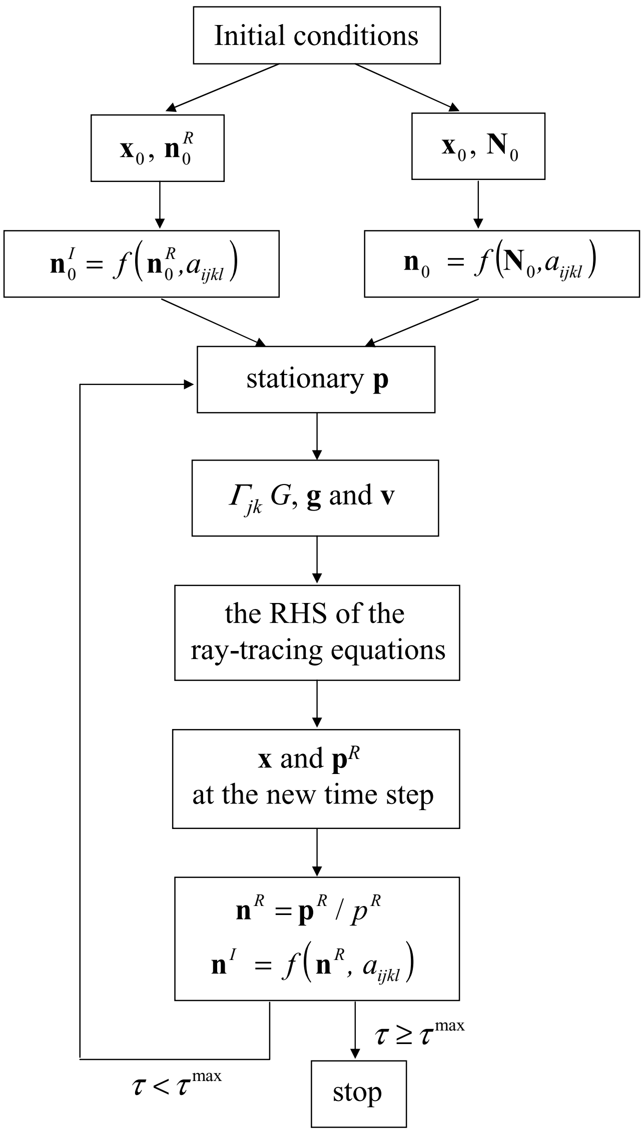

Figure 1. A scheme of the numerical procedure for ray tracing in anisotropic viscoelastic media.

(2) We calculate the initial slowness vector $\mathbf{p}_0$. This can be done iteratively for both types of initial conditions.

(a) If the wave normal $\mathbf{n}_0^{\mathrm{R}}$ is specified, we seek two real-valued angles defining the direction of $\mathbf{n}_0^{\mathrm{I}}$. From the initial guess of $\mathbf{n}_0^{\mathrm{I}}$, we calculate the complex-valued direction $\mathbf{n}_0 = \mathbf{n}_0^{\mathrm{R}} + \mathrm{i}\ \mathbf{n}_0^{\mathrm{I}}$, the Christoffel tensor $\Gamma_{jk}(\mathbf{n}_0) = a_{ijkl} n_{0i} n_{0l}$, its eigenvalues G and eigenvectors $\mathbf{g}$ and subsequently the complex phase velocity c, $c = \left[a_{ijkl} n_{0i} n_{0l} g_j g_k\right]^{\frac{1}{2}}$, and complex slowness p, $p = 1/c$. Then, we calculate complex energy velocity vector $\mathbf{v}$, $v_i = a_{ijkl} p_l g_j g_k$, and the ray direction $\mathbf{N} = \mathbf{v}/v$. Vector $\mathbf{N}$ is generally complex valued. Therefore, we have to vary, iteratively, $\mathbf{n}_0^{\mathrm{I}}$ to minimize the imaginary part of $\mathbf{N}$ to obtain the real-valued $\mathbf{N}$. As the misfit function, we can use the non-negative scalar function, $f = \mathbf{N}^{\mathrm{I}} \cdot \mathbf{N}^{\mathrm{I}}$, which must be zero at the stationary point.

(b) If the initial ray direction $\mathbf{N}_0$ is specified, we use iterations to find four real-valued angles, defining the real and imaginary parts of slowness direction $\mathbf{n}_0$. We can adopt the ray direction $\mathbf{N}_0$ as the initial guess of $\mathbf{n}_0$ and proceed in an analogous way to the previous case. The misfit function can be defined as the modulus of the complex-valued deviation between the fixed and predicted ray vectors. Alternatively, instead of iterations, we can solve a system of coupled algebraic equations of the 6th order in three unknowns, p_1, p_2 and p_3 (see Vavryčuk 2006, 2007b).

(3) Finding the stationary slowness vector $\mathbf{p}$, we calculate the Christoffel tensor $\Gamma_{jk}(\mathbf{p})$, its eigenvalues G and eigenvectors $\mathbf{g}$ and the complex energy velocity vector $\mathbf{v}$.

(4) We evaluate the right-hand sides of the ray tracing eqs (29) and (35)

$$R_i^{(1)} = N_i \frac{v^{\mathrm{R}} v^{\mathrm{R}} + v^{\mathrm{I}} v^{\mathrm{I}}}{v^{\mathrm{R}}}, \tag{42}$$

$$R_i^{(2)} = -\frac{1}{2}\left[\left(\frac{\partial a_{jklm}}{\partial x_i} p_j p_m g_k g_l\right)^{\mathrm{R}} + \frac{v^{\mathrm{I}}}{v^{\mathrm{R}}}\left(\frac{\partial a_{jklm}}{\partial x_i} p_j p_m g_k g_l\right)^{\mathrm{I}}\right]. \tag{43}$$

(5) We move forward along a ray by one time step $\Delta\tau^{\mathrm{R}}$

$$x_i\left(t_0 + \Delta\tau^{\mathrm{R}}\right) = x_i(t_0) + \Delta x_i, \ \ p_i^{\mathrm{R}}\left(t_0 + \Delta\tau^{\mathrm{R}}\right) = p_i^{\mathrm{R}}(t_0) + \Delta p_i^{\mathrm{R}}, \tag{44}$$

where

$$\Delta x_i = R_i^{(1)} \Delta\tau^{\mathrm{R}}, \ \ \Delta p_i^{\mathrm{R}} = R_i^{(2)} \Delta\tau^{\mathrm{R}}. \tag{45}$$

(6) Normalizing $\mathbf{p}^{\mathrm{R}}\left(t_0 + \Delta\tau^{\mathrm{R}}\right)$, we obtain $\mathbf{n}^{\mathrm{R}}\left(t_0 + \Delta\tau^{\mathrm{R}}\right)$ and calculate $\mathbf{n}^{\mathrm{I}}\left(t_0 + \Delta\tau^{\mathrm{R}}\right)$ in a way analogous to (2a). Afterwards, we again continue by following steps (3)–(6). The whole process is repeated until the traveltime reaches a predefined maximum value.

Tracing rays in isotropic viscoelastic media is much simpler than described in the above scheme because the stationary slowness vector is homogeneous. Vector $\mathbf{p}^{\mathrm{I}}$ can readily be obtained from ray direction $\mathbf{N}$ and attenuation A

$$\mathbf{p}^{\mathrm{I}} = \mathbf{N}\, A, \quad A = \left(c^{-1}\right)^{\mathrm{I}}. \tag{46}$$

Hence no iterative procedure for calculating $\mathbf{p}^{\mathrm{I}}$ is needed and solving the ray tracing equations in viscoelasticity is quite analogous to solving them in elasticity.

7 NUMERICAL EXAMPLES

In this section, I examine the behaviour of rays in viscoelastic media numerically. I consider the *SH* wave propagating in two vertically inhomogeneous isotropic media (models A and B) and in two vertically inhomogeneous transversely isotropic media, with the vertical axis of symmetry (models C and D). The frequency of the signal is assumed to be 20 Hz. The models are defined by the depth-dependent density-normalized viscoelastic parameters a_{44} and a_{66}:

$$a_{44}(z) = \left(a_{44}^0\right)^{\mathrm{R}}\left(1 + \varepsilon_{44}^{\mathrm{R}} z\right)^2 + \mathrm{i}\left(a_{44}^0\right)^{\mathrm{I}}\left(1 + \varepsilon_{44}^{\mathrm{I}} z\right)^2, \tag{47}$$

$$a_{66}(z) = \left(a_{66}^0\right)^{\mathrm{R}}\left(1 + \varepsilon_{66}^{\mathrm{R}} z\right)^2 + \mathrm{i}\left(a_{66}^0\right)^{\mathrm{I}}\left(1 + \varepsilon_{66}^{\mathrm{I}} z\right)^2, \tag{48}$$

where a_{44}^0 and a_{66}^0 are the values of a_{44} and a_{66} at the surface ($z = 0$). The real parts of a_{44} and a_{66} describe the relevant elastic (non-attenuating) media. The imaginary parts of a_{44} and a_{66} describe attenuation. Models A and C display constant Q with depth and models B and D display increasing Q with depth. The parameters defining the models are summarized in Table 1. The transversely isotropic models are characterized by 22 per cent of velocity anisotropy (calculated for the elastic background) and by 67 per cent

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Table 1. A summary of the parameters describing the models.

Model	Type	$\left(a_{44}^0\right)^{\mathrm{R}}$ (km^{2}s^{-2})	$\left(a_{66}^0\right)^{\mathrm{R}}$ (km^{2}s^{-2})	Q_{44}^0	Q_{66}^0	a_V^0 (per cent)	a_Q^0 (per cent)	ε^{R}	ε^{I}
A	I	6.25	6.25	10	10	0	0	−0.020	−0.020
B	I	6.25	6.25	10	10	0	0	−0.020	0.025
C	VTI	6.25	4.00	10	5	22	67	−0.020	−0.020
D	VTI	6.25	4.00	10	5	22	67	−0.020	0.025

Note: Types I/VTI mean isotropy/transverse isotropy with a vertical axis of symmetry, a_V^0 is the phase velocity anisotropy in the elastic background medium, and it is calculated as $a_V^0 = 200\left(V_Z^0 - V_H^0\right)/\left(V_Z^0 + V_H^0\right)$, where $V_Z^0 = \sqrt{\left(a_{44}^0\right)^{\mathrm{R}}}$, $V_H^0 = \sqrt{\left(a_{66}^0\right)^{\mathrm{R}}}$, a_Q^0 is the Q-factor anisotropy, $a_Q^0 = 200\left(Q_{44}^0 - Q_{66}^0\right)/\left(Q_{44}^0 + Q_{66}^0\right)$. The velocity anisotropy and the Q-factor anisotropy do not change with depth. Quantities ε^{R} and ε^{I} denote the real and imaginary parts of the velocity gradient in the medium.

Q-factor anisotropy. All calculations are performed in attenuating, as well as non-attenuating media.

Figs 2(a) and (c) show the phase velocities as a function of depth, for the models under study. The velocities are calculated at the elastic background medium. The velocity gradient is constant and falls within typical values observed in the Earth's crust. Figs 2(b) and (d) show the Q-factor as a function of depth. At the surface, the Q-factor is 10 for the isotropic medium and $Q_{44} = 10$ and $Q_{66} = 5$ for the transversely isotropic medium. The other values of the Q-matrix are not needed for the propagation of the *SH* waves. The values of Q at the surface are rather extreme and observed, exceptionally, for some very soft sedimentary rocks. With increasing depth, the value of Q is constant for models A and C but increases for models B and D. The depth-independent Q is unrealistic, and

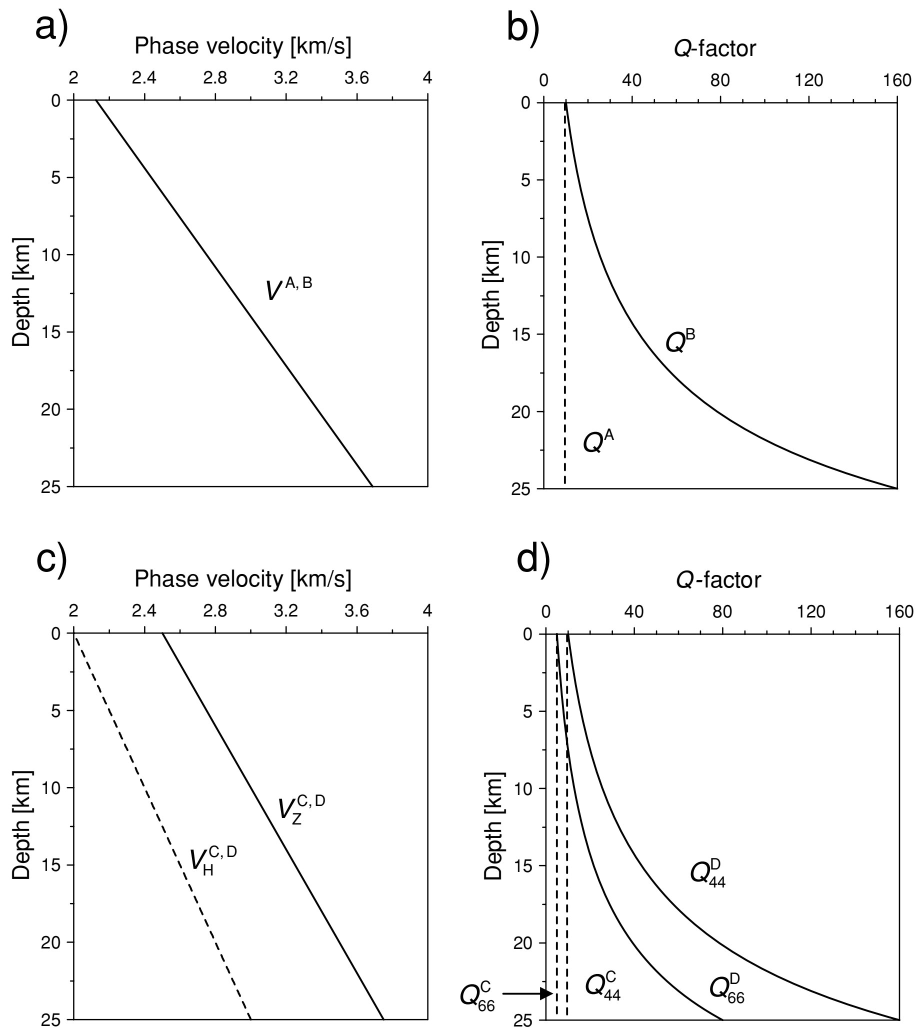

Figure 2. *SH* phase velocities and Q-factors as a function of depth. (a) Phase velocity in the elastic background of models A and B, (b) Q-factors for model A (dashed line) and model B (solid line), (c) phase velocity in the elastic background of models C and D and (d) Q-factors for model C (dashed line) and model D (solid line). V_H, the horizontal phase velocity and V_Z, the vertical phase velocity.

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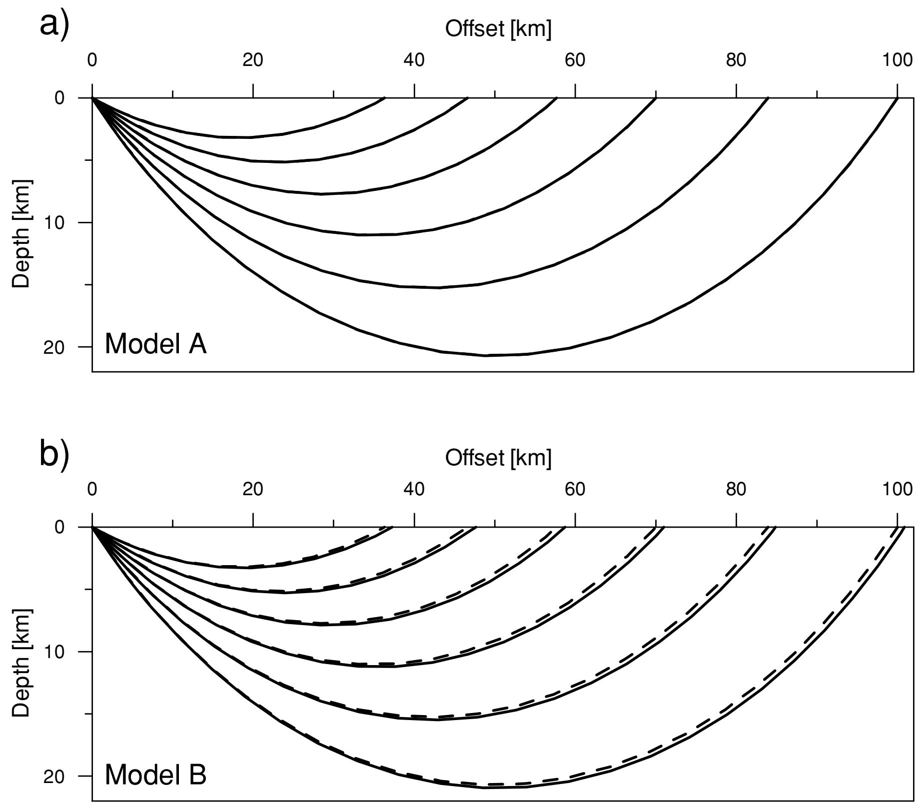

Figure 3. Ray fields in models A (a) and B (b). Solid line, rays calculated in the viscoelastic medium; dashed line, rays calculated in the elastic background. For parameters of the media and rays, see the text.

here it is used just to demonstrate properties of ray fields for such specific attenuation models. In models B and D, the value of Q increases non-linearly to $Q_{44} = 160$ and $Q_{66} = 90$ at depth of 25 km.

Fig. 3 shows ray fields in models A and B, generated by a point source situated at the surface. The rays are calculated by package Matlab, using the solver for a system of ordinary differential equations ODE45. The time step is adaptive attaining values from 10^{-5} s up to 1 s. The rays are shot along a profile in the x-z plane, with incidences between 20° and 45° with step of 5°. The incidence angles are measured downwards from the horizontal axis. The ray fields are shown for elastic (dashed line), as well as viscoelastic (full line) media. For model A, both ray fields coincide. For model B, the differences in ray fields are visible, but they are small and rather marginal. The only significant differences can be observed in traveltimes, provided the signal is of sufficiently high frequency (see Fig. 4). This means a frequency of about 20 Hz or higher in the presented numerical experiment. The differences are negative meaning that the traveltimes in media with attenuation are less than those in media without attenuation, hence the waves propagate faster in media with attenuation (provided the elastic background is the same in both models). As expected, the differences are more pronounced in model A, which displays high attenuation at all depths. For model B, the differences in traveltimes are observed for shallow rays, which sample a highly attenuating structure. The deeper rays sample mainly a structure with low attenuation, hence the differences are less significant. As a consequence, the differences between elastic and viscoelastic traveltimes decrease with an increasing offset for receivers, with offset larger than 45 km.

Fig. 5 shows the ray fields generated by a point source in model C and D. The rays are shot with incidences between 20° and 50°, with step of 5°. In both models, the ray field in a medium with attenuation is different from that in a medium without attenuation. Hence, in anisotropic medium, a constant Q-model does not necessarily imply the ray field being identical with that in the elastic background. The differences in the ray geometry of elastic and viscoelastic rays in models C and D are remarkably larger than those in models A and B and reflect the fact that models C and D are more attenuating. This is pronounced also in differences in traveltimes (Fig. 6). The differences between elastic and viscoelastic traveltimes as a function of the epicentral distance have a similar shape in models C and D compared with models A and B, but the scale is different. The differences are about four times larger for models C and D than for models A and B.

8 DISCUSSION: REAL VERSUS COMPLEX RAYS

So far, it has been assumed that ray theory in viscoelasticity must necessarily deal with complex rays (Hearn & Krebes 1990a,b; Le *et al.* 1994; Thomson 1997; Chapman *et al.* 1999; Kravtsov *et al.* 1999; Hanyga & Seredyňska 2000). Theory of complex rays, however, encounters essential difficulties. First of all, it is not clear in complex ray theory how to build a medium model in the complex space from a model in the real space which describes a 3-D inhomogeneous attenuating structure with interfaces. On the contrary, the presented approach shows that viscoelastic ray theory with real-valued rays

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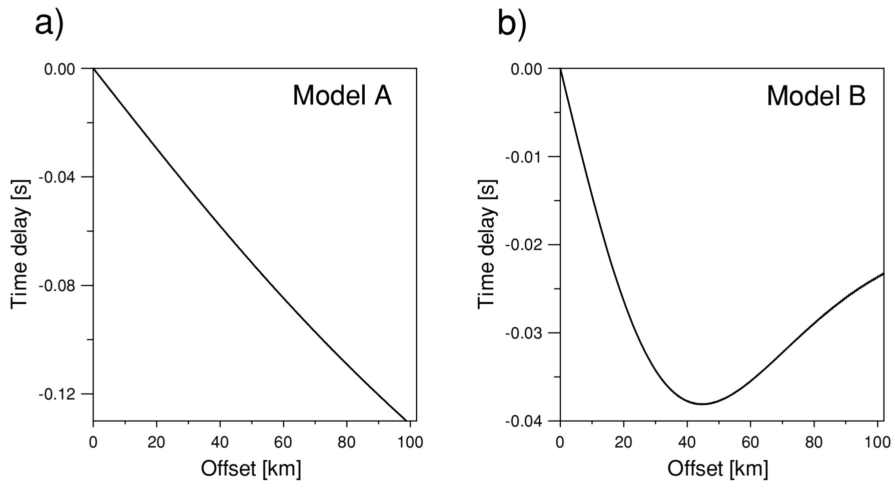

Figure 4. Differences in traveltimes of waves propagating in the attenuating medium and in the elastic background. (a) traveltime differences in model A and (b) traveltime differences in model B.

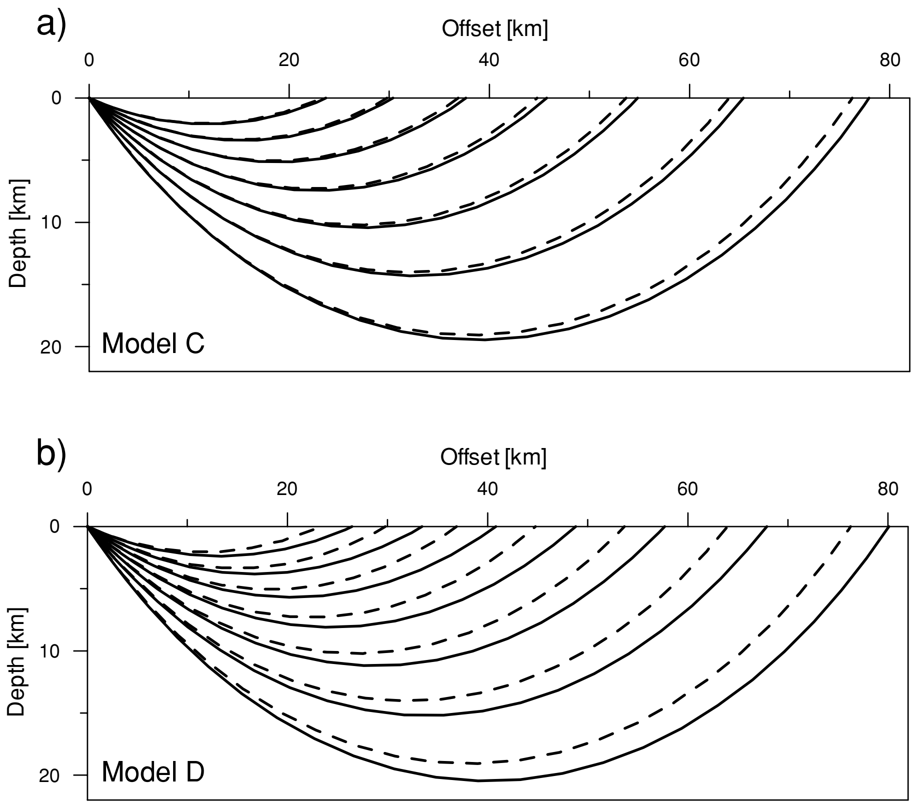

Figure 5. Ray fields in models C (a) and D (b). Solid line, rays calculated in the viscoelastic medium; dashed line; rays calculated in the elastic background. For parameters of the media and rays, see the text.

can be developed. The approach is based on the assumption that rays are trajectories in the real space, characterized by the stationary real traveltime. The imaginary part of the traveltime reflects attenuation along a ray. Attenuation in directions perpendicular to a ray is neglected in the phase term but included in the ray amplitude $\mathbf{U}(\mathbf{x})$. As a consequence, the eikonal equation, as the partial differential equation for the complex-valued traveltime, is not satisfied exactly but only approximately.

The existence of real rays does not imply that the slowness vector or other ray quantities must be real valued. In viscoelastic media, the slowness vector is always complex valued. In anisotropic media, the slowness vector is inhomogeneous. In isotropic media, the slowness vector is homogeneous. This implies that inhomogeneous waves (described by the inhomogeneous slowness vector) cannot be accurately treated in isotropic media by the standard geometrical ray theory. Varying amplitude along the wave front of inhomogeneous waves is projected into spatially-dependent ray amplitude, see eq. (7). But the phase must always produce a homogeneous slowness vector, see eq. (10). This fact is well known for ray theory in elasticity and remains valid also for real ray theory in viscoelasticity.

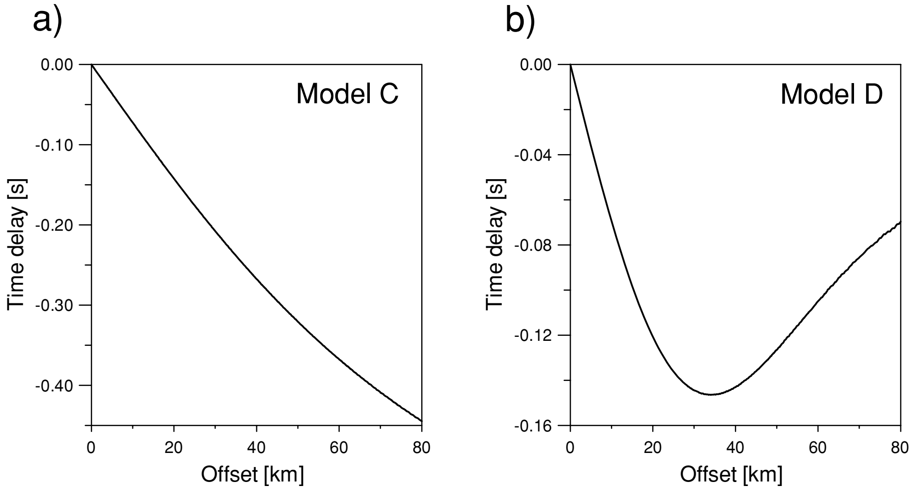

Figure 6. Differences in traveltimes of waves propagating in the attenuating medium and in the elastic background. (a) traveltime differences in model C and (b) traveltime differences in model D.

The reason for this seemingly surprising limitation is that inhomogeneous waves in isotropic media are not waves with the stationary phase. Therefore, they cannot be described exactly but only approximately using the zero-order ray theory. Consequently, the exact solution derived, for example, for the reflection/transmission of plane inhomogeneous waves at interfaces in isotropic attenuating media (Wennerberg 1985; Winterstein 1987; Caviglia & Morro 1992; Carcione 2007) differs from the ray-theoretical solution, which is approximate. In modelling of inhomogeneous waves, a higher accuracy can be achieved if higher-order ray approximations are incorporated (see Vavryčuk 2007c), or if the ray theory is in some way extended or modified. The same applies to inhomogeneous waves in anisotropic viscoelastic media, when the wave inhomogeneity is different from that consistent with the stationary phase.

9 CONCLUSIONS

The real ray tracing in anisotropic viscoelastic media displays substantial differences compared with elastic media. The rays are frequency-dependent and the ray fields must be calculated for each frequency separately. The proposed ray tracing equations produce real-valued vectors $\mathbf{x}$ and $\mathbf{p}^R$. Vector $\mathbf{p}^I$ is calculated, independent of the ray tracing equations, at each time step. It is computed by iterations from the condition that the complex energy velocity vector $\mathbf{v}$ is homogeneous. Several iterations are usually sufficient for finding of $\mathbf{p}^I$, but still these additional calculations slow down the ray tracing procedure. For isotropic media, the problem simplifies because the slowness vector $\mathbf{p}$ is homogeneous, and the requirements on the computer time are essentially the same as in elastic media.

Using numerical examples, it has been shown that the ray fields are not very sensitive to attenuation of the medium. The ray fields in weakly attenuating media (Q higher than 30) are almost indistinguishable from those in elastic media. In isotropic media with constant Q-factor, the rays are exactly identical with those in elastic media. This applies to all values of constant Q. In anisotropic media, this property is lost. For moderately attenuating (Q between 5–20) anisotropic or isotropic media with varying Q, the differences in ray fields in attenuating and non-attenuating media can be visible and significant. Observing the differences in traveltimes in attenuating and non-attenuating media depend on several factors. They depend: first, on how strongly attenuating the medium is; second, on the length of the ray along which the wave propagates and third, on the predominant frequency of the wavefield. In general, the differences increase with magnitude of attenuation and with length of the ray, and they are better distinguishable for high frequencies.

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APPENDIX: HOMOGENEITY OF THE COMPLEX ENERGY VELOCITY VECTOR

The complex energy velocity vector $\mathbf{v}$ is defined as

$$v_i = \frac{dx_i}{d\tau}, \tag{A1}$$

where $\mathbf{x}$ is the position vector along a ray and τ is the complex-valued traveltime. Hence,

$$dx_i = v_i d\tau = v_i^R d\tau^R - v_i^I d\tau^I + i\left(v_i^R d\tau^I + v_i^I d\tau^R\right). \tag{A2}$$

Imposing the condition that the ray is a trajectory in the real space, vector $d\mathbf{x}$ is real valued; so, we get

$$v_i^R d\tau^I + v_i^I d\tau^R = 0. \tag{A3}$$

Consequently,

$$v_i^I = -v_i^R \frac{d\tau^I}{d\tau^R}, \tag{A4}$$

which implies that $\mathbf{v}^R$ and $\mathbf{v}^I$ are parallel. Introducing the direction vector $\mathbf{N}$,

$$v_i^R = N_i v^R, \quad v_i^I = N_i v^I, \tag{A5}$$

we obtain,

$$v_i = N_i\left(v^R + i v^I\right) = N_i v, \tag{A6}$$

where $\mathbf{N}$ is the real-valued vector called the ray direction and $\mathbf{v}$ is the complex energy velocity. Since ray direction $\mathbf{N}$ is real valued, $\mathbf{v}$ must be a homogeneous vector.

From eqs (A4) and (A5) we further obtain,

$$\frac{d\tau^I}{d\tau^R} = -\frac{\mathbf{v}^I}{\mathbf{v}^R}. \tag{A7}$$

Consequently,

$$\tau^I = -\int_A^B \frac{v^I}{v^R} d\tau^R, \tag{A8}$$

where A and B denote the start and endpoints of a ray.