ASYMPTOTIC ELASTODYNAMIC GREEN FUNCTION IN THE KISS SINGULARITY IN HOMOGENEOUS ANISOTROPIC SOLIDS

VÁCLAV VAVRYČUK

Geophysical Institute, Acad. Sci. Czech Rep., Prague, Czech Republic *

ABSTRACT

The far-field asymptotic formula is derived for the elastodynamic Green function in the kiss singularity in homogeneous anisotropic solids. In contrast to standard asymptotics in regular directions the derived formula is more complex and expressed in the form of a 1-D integral. This integral is specified for the kiss singularity along the symmetry axis in transverse isotropy and along the fourfold symmetry axes in tetragonal and cubic symmetries. The shape of the slowness surface in the singularity is regular in transverse isotropy and the amplitude of the Green function is expressed by means of the Gaussian curvature of this surface in the singularity. However, the shape of the slowness surface is irregular and the Gaussian curvature is not defined in the singularity in tetragonal or cubic symmetries. In this case, the amplitude of the Green function is expressed by means of the generalized Gaussian curvature.

1. INTRODUCTION

Three waves propagate in homogeneous anisotropic media having, in general, three different phase velocities and polarizations. The phase velocities and polarization vectors are determined as eigenvalues and eigenvectors of the Christoffel tensor. For some directions, the Christoffel tensor can degenerate, and two waves (or even all three waves) then propagate with the same phase velocity. These directions of degeneracy can cause anomalies in the shape of the slowness surface as well as singularities in the field of polarization vectors (see *Alshits, Sarychev and Shuvalov, 1985; Shuvalov and Every, 1997; Shuvalov, 1998*). One of the important types of degeneracy is the direction in which the slowness sheets of two waves touch tangentially. If this direction is isolated and the field of polarization vectors is singular, we call it the "kiss singularity" (*Crampin and Yedlin, 1981*). The simplest example of the kiss singularity is the symmetry axis in transversely isotropic solids (*Vavryčuk, 1999*). This singularity appears on the *S*-wave slowness sheets and displays a simple pattern of polarization vectors (see Fig. 1). The topological charge of the field of polarization vectors in the singularity equals 1 (*Shuvalov, 1998*). The slowness sheets of the degenerate waves are differentiable at any order, hence the shape of the slowness sheets is regular in this singularity. However, the kiss singularity in other kinds of anisotropy can be more complex. The topological charge of the field of polarization vectors in the singularity can attain values 0, 1 (see Fig. 1) or

* Boční II/1401, 141 31 Praha 4, Czech Republic (e-mail: vv@ig.cas.cz)

–1 (see Fig. 2), and the shape of the slowness surface can display irregularities in the singularity. For example, the slowness surface can be differentiated at the first order only and the Gaussian curvature of the slowness surface may not be defined in the singularity (*Shuvalov and Every, 1996*).

The anomalous behaviour of the slowness surface and polarization vectors near singularities causes that the standard far-field asymptotics for the Green function (*Buchwald, 1959; Lighthill, 1960; Burridge, 1967; Yeatts, 1984; Every and Kim, 1994; Červený, 2001*) can not be applied, but a modified approach is required. The far-field asymptotics for the simplest type of the singularity, the kiss singularity in transversely isotropic solids, has been derived by *Vavryčuk (1997, 1999)* who showed that the asymptotics is significantly affected by the coupling of the degenerate waves in the kiss singularity and its vicinity. The asymptotic Green function in transversely isotropic solids has also been studied by *Gridin (2000)*. A general mathematical approach to calculating the asymptotic Green functions in kiss singularities in all kinds of anisotropy with examples in transversely isotropic solids is outlined by *Borovikov and Gridin (2001)*. In this paper, the far-field asymptotics in the kiss singularity in general anisotropy is studied and specified for the kiss singularity along a rotational symmetry axis in transverse isotropy and along fourfold symmetry axes in tetragonal and cubic symmetries. The asymptotics is studied in the singularity itself; directions close to the singularity are not considered. The kiss singularity in cubic or tetragonal symmetries is of particular interest, because the shape of the slowness surface is irregular in the singularity. Consequently, the Gaussian curvature is not uniquely defined in the singularity, but depends on the direction from which the singularity is approached (*Shuvalov and Every, 1996*). The far-field asymptotics derived will be applicable even for the singularities that are touched by parabolic lines on the slowness surface and by caustics on the wave surface (see Fig. 3).

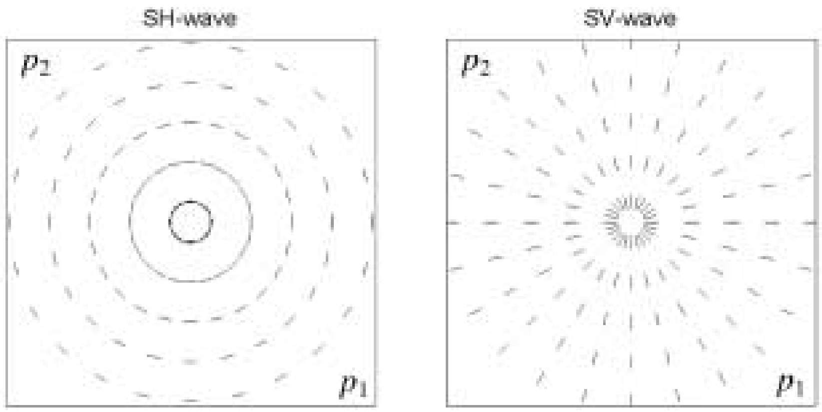

Fig. 1. Singular behaviour of polarization vectors on the slowness surface near the kiss singularity in transverse isotropy. The topological charge is +1. Symbols p_1 and p_2 denote the Cartesian coordinate axes perpendicular to the direction of the singularity.

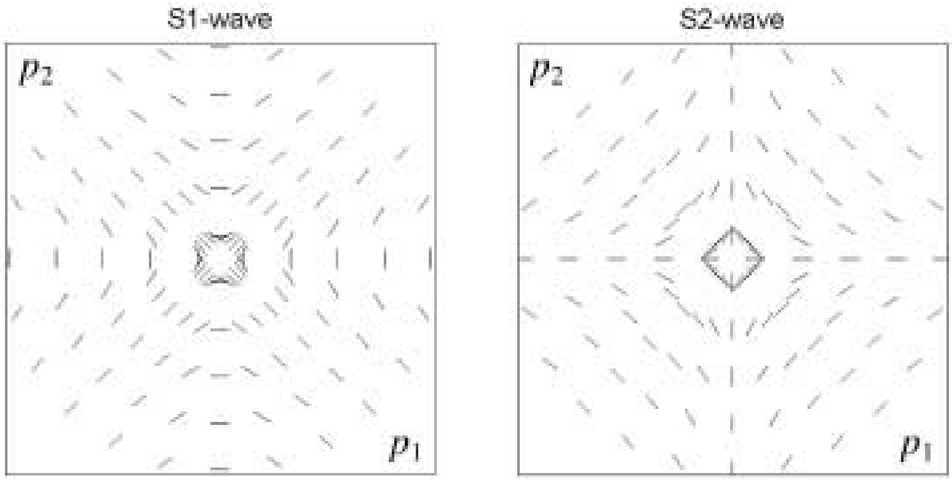

Fig. 2. Singular behaviour of polarization vectors on the slowness surface near the fourfold symmetry axis in tetragonal anisotropy. The topological charge is –1. Elastic parameters of the anisotropy are (in km^2s^{-2}): $a_{11} = a_{22} = 6.25$, $a_{33} = 9.38$, $a_{12} = 2.71$, $a_{13} = a_{23} = 3.13$, $a_{66} = 2.08$, $a_{44} = a_{55} = 2.92$. Symbols p_1 and p_2 denote the Cartesian coordinate axes perpendicular to the direction of the singularity.

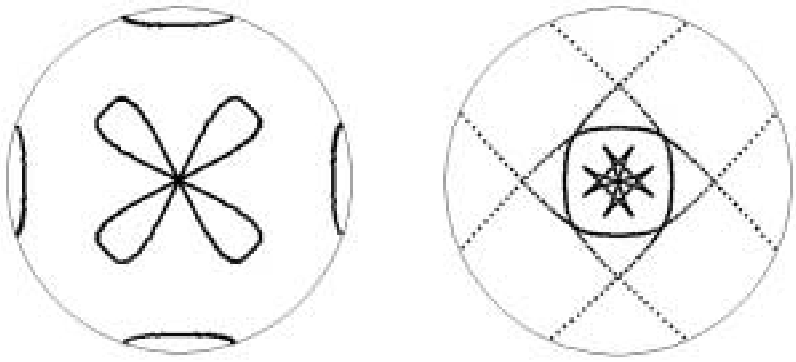

Fig. 3. Parabolic lines (left) and caustics (right) for the slow *S* wave near the kiss singularity in cubic anisotropy. The elastic parameters are (in km^2s^{-2}): $a_{11} = a_{22} = a_{33} = 6.25$, $a_{12} = a_{13} = a_{23} = 3.47$, $a_{44} = a_{55} = a_{66} = 2.08$. The singularity is along the vertical axis which coincides with the centre of the circles. The circles correspond to the deviation 13° of the wave normals (left) or ray directions (right) from the vertical axis. Equal-are projection is used *(Aki and Richards, 1980, Eq. 4.17)*. Parabolic lines and caustics separate convex and saddle-shaped areas on the slowness and wave surfaces.

2. ASYMPTOTIC FORMULA FOR THE GREEN FUNCTION

The exact elastodynamic Green function $G_{kl}(\mathbf{x},t)$ in unbounded, homogeneous, anisotropic and perfectly elastic media can be expressed as follows (*Burridge, 1967, Eq. 4.6; Wang and Achenbach, 1994, Eq. 13*):

$$G_{kl}(\mathbf{x},t) = -\frac{H(t)}{8\pi^2\rho} \sum_{m=1}^{3} \int_{S(\mathbf{n})} \frac{g_k^m g_l^m}{\left(c^m\right)^3} \dot{\delta}\left(t - \frac{\mathbf{n}\cdot\mathbf{x}}{c^m}\right) dS(\mathbf{n}) \quad . \tag{1}$$

Superscript $m = 1,2,3$ denotes the type of wave (P, $S1$ and $S2$), $\mathbf{g} = \mathbf{g}(\mathbf{n})$ is the unit polarization vector, $c = c(\mathbf{n})$ is the phase velocity, ρ is the density of the medium, t is time, $H(t)$ is the Heaviside step function, $\dot{\delta}(t)$ is the time derivative of the Dirac delta function, $\mathbf{x}$ is the position vector of the observation point, and $\mathbf{n}$ is the wave normal. The integration is over unit sphere $S(\mathbf{n})$. If the integration is performed over the slowness surface, the integral takes the following form *(Burridge, 1967, Eq. 5.4)*:

$$G_{kl}(\mathbf{x},t) = -\frac{H(t)}{8\pi^2\rho} \sum_{m=1}^{3} \int_{S(\mathbf{p})} \frac{g_k^m g_l^m}{v^m} \dot{\delta}(t - \mathbf{p}\cdot\mathbf{x})\, dS(\mathbf{p}) \ , \tag{2}$$

where $v = |v_i| = |a_{ijkl}\, p_l g_j g_k|$ is the group velocity of the wave (see *Červený, 2001, Eq. 2.4.46*), $\mathbf{p} = \mathbf{n}/c$ is the slowness vector, a_{ijkl} is the density-normalized elasticity tensor, and $S(\mathbf{p})$ is the slowness surface. Both equations (1) and (2) formally contain a surface integral, but actually the Dirac delta function in the integrand reduces the surface integral to a line integral. The integration line in (2) is an intersection of the slowness surface with a plane whose normal is parallel to $\mathbf{x}$, and which moves with time from the source towards the observation point.

The evaluation of (1) and (2) is complicated and can be performed analytically only for a few cases of very simple anisotropy (*Payton, 1983; Burridge, Chadwick and Norris, 1993; Gridin, 2000; Vavryčuk, 2001*). For general anisotropy, it can be evaluated either numerically (*Wang and Achenbach, 1994*), or asymptotically (*Every and Kim, 1994*). The asymptotic Green function is much simpler than the exact Green function because it is significant only at times close to arrival time t_0 for which the integration line collapses into a stationary point $\mathbf{p}_0$. The far-field asymptotic Green function $G_{kl}^{far}(\mathbf{x},t)$ for one particular wave is expressed as follows:

$$G_{kl}^{far}(\mathbf{x},t) = -\frac{H(t)}{8\pi^2\rho} \int_{S^{\varepsilon}} \frac{g_k g_l}{v} \dot{\delta}\left(t - \mathbf{p}_0\cdot\mathbf{x} + \frac{1}{2}|\mathbf{x}|\, \mathbf{s}^T \overline{\mathbf{K}}\, \mathbf{s}\right) ds_1 ds_2 \quad , \tag{3}$$

where $\mathbf{x}$ is parallel to the normal of the slowness surface at $\mathbf{p}_0$, vector $\mathbf{s} = (s_1, s_2)^T$ defines the position of $\mathbf{p}$ in a local coordinate system, and $\overline{\mathbf{K}} = \overline{\mathbf{K}}(s_1, s_2)$ is a 2×2 matrix.

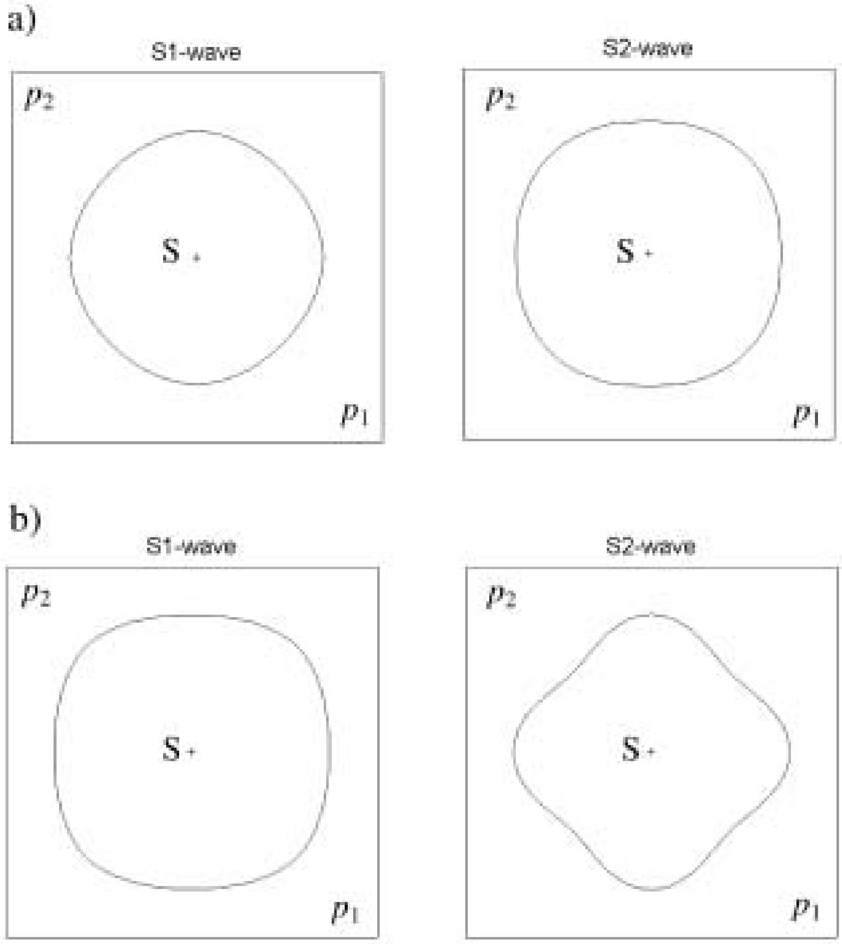

Fig. 4. The shape of the integration line for the kiss singularity in two tetragonal solids. Point *S* denotes the singularity in the direction $\mathbf{n} = (0,0,1)^T$. The elastic parameters are (in $km^2 s^{-2}$): (a) $a_{11} = a_{22} = 6.25$, $a_{33} = 9.38$, $a_{12} = 2.71$, $a_{13} = a_{23} = 2.35$, $a_{66} = 2.08$, $a_{44} = a_{55} = 2.92$, and (b) $a_{11} = a_{22} = 6.25$, $a_{33} = 9.38$, $a_{12} = 2.71$, $a_{13} = a_{23} = 3.13$, $a_{66} = 2.08$, $a_{44} = a_{55} = 2.92$. The topological charge is +1 for (a) and –1 for (b). Symbols p_1 and p_2 denote the Cartesian coordinate axes perpendicular to the direction of the singularity.

Local coordinates s_1 and s_2 are perpendicular to $\mathbf{x}$ and have their origin at $\mathbf{p}_0$. The integration is over small neighbourhood S^ε of point $\mathbf{p}_0$ of slowness surface $S(\mathbf{p})$. If the slowness surface is regular at $\mathbf{p}_0$, the matrix $\bar{\mathbf{K}}$ is independent of s_1 and s_2 and coincides with the standard curvature matrix $\mathbf{K}$ of the slowness surface at stationary point $\mathbf{p}_0$ (see *Burridge, 1967, Eq. 6.7*). The integration curve near stationary point $\mathbf{p}_0$ is an ellipse (or a circle) for convex or concave surfaces and a hyperbola for saddle-shaped surfaces. If the slowness surface is irregular at $\mathbf{p}_0$, but regular for all points in the vicinity of $\mathbf{p}_0$, matrix $\bar{\mathbf{K}}$ is defined using the values of standard curvature matrix $\mathbf{K}$ in the immediate vicinity of $\mathbf{p}_0$:

$$\bar{\mathbf{K}}(\varphi) = \lim_{s \to 0} \mathbf{K}(s, \varphi) \ , \tag{4}$$

where

$$s = \sqrt{s_1^2 + s_2^2} \quad , \quad \varphi = \operatorname{atan} \frac{s_2}{s_1} \ . \tag{5}$$

Obviously, the integration curve in (3) is no longer an ellipse or hyperbola but its form is more complex (see Fig. 4).

3. REGULAR DIRECTION

Let us consider a regular field of polarization vectors near $\mathbf{p}_0$ and a regular convex shape of the slowness surface at $\mathbf{p}_0$. Since $\mathbf{g} = \mathbf{g}(s_1, s_2)$ and $v = v(s_1, s_2)$ in (3) are continuous functions at $\mathbf{p}_0$, we can apply the properties of the δ-function and bring $\mathbf{g}$ and v in front of the integral,

$$G_{kl}^{far}(\mathbf{x}, t) = -\frac{H(t)}{8\pi^2 \rho} \frac{g_k(\mathbf{p}_0) g_l(\mathbf{p}_0)}{v(\mathbf{p}_0)} \int_{S^\varepsilon} \dot{\delta}\left(t - \mathbf{p}_0 \cdot \mathbf{x} + \frac{1}{2}|\mathbf{x}| \mathbf{s}^T \bar{\mathbf{K}} \ \mathbf{s}\right) ds_1 ds_2 \ . \tag{6}$$

Since the slowness surface is convex at $\mathbf{p}_0$, the integration line in (6) is an ellipse. Hence we can choose coordinates s_1 and s_2 parallel to the directions of principal curvatures K_{11} and K_{22} at $\mathbf{p}_0$:

$$\mathbf{s}^T \mathbf{K} \mathbf{s} = K_{11} s_1^2 + K_{22} s_2^2 \ , \tag{7}$$

and substitute coordinates s_1 and s_2 by coordinates u and σ

$$u = \frac{1}{2}|\mathbf{x}| \left(K_{11} s_1^2 + K_{22} s_2^2\right) \ , \quad \sigma = \operatorname{atan} \frac{\sqrt{K_{22}} s_2}{\sqrt{K_{11}} s_1} \ ,$$

$$ds_1 ds_2 = \frac{1}{\sqrt{K_{11}K_{22}}} \frac{1}{|\mathbf{x}|} du d\sigma \ . \tag{8}$$

We obtain

$$G_{kl}^{far}(\mathbf{x},t) = -\frac{H(t)}{8\pi^2 \rho} \frac{g_k(\mathbf{p}_0) g_l(\mathbf{p}_0)}{v(\mathbf{p}_0)} \frac{1}{\sqrt{K_{11}K_{22}}} \frac{1}{|\mathbf{x}|} \times$$

$$\int \dot{\delta}\left(t - \mathbf{p}_0 \cdot \mathbf{x} + u\right) du \int_0^{2\pi} d\sigma \ . \tag{9}$$

Finally, the asymptotic Green function reads *(Burridge, 1967, Eq. 6.8)*

$$\boxed{G_{kl}^{far}(\mathbf{x},t) = \frac{1}{4\pi\rho} \frac{g_k g_l}{v\sqrt{K}} \frac{\delta\left(t - \mathbf{p}_0 \cdot \mathbf{x}\right)}{|\mathbf{x}|} \ ,} \tag{10}$$

where $K = K_{11}K_{22}$ is the Gaussian curvature of the convex slowness surface, $\mathbf{g}$ is the polarization vector, and v is the group velocity. All the quantities are evaluated at $\mathbf{p}_0$. A similar formula to (10) can also be written for the concave or saddle-shaped slowness surfaces at $\mathbf{p}_0$. For the concave surface, an additional minus sign appears in the formula, and for the saddle-shaped surface, instead of the Gaussian curvature its absolute value is used in the formula and the δ-function in (10) is replaced by its Hilbert transform (*Burridge, 1967, Eq. 6.9; Every and Kim, 1994*).

4. KISS SINGULARITY IN TRANSVERSE ISOTROPY

The kiss singularity appears on the *S*-wave slowness sheets and coincides with the symmetry axis of transverse isotropy. One of the *S* waves has a polarization in the plane perpendicular to the symmetry axis, and is denoted as the *SH* wave. The other *S* wave has the polarization in the plane defined by the slowness vector and the symmetry axis, and is denoted as the *SV* wave (see Fig. 1). The topological charge of the field of polarization vectors in the singularity is 1 (*Darinskii, 1994; Shuvalov, 1998*). The shape of the slowness surface is regular in the singularity. The curvature matrix $\mathbf{K}$ of each *S* wave is diagonal in the singularity, and the principal curvatures K_{11} and K_{22} are equal,

$$K_{11} = K_{22} = \sqrt{K} \ , \tag{11}$$

where K denotes the Gaussian curvature in the singularity for the respective *S* wave. Equation (3) takes the following form:

$$G_{kl}^{far}(\mathbf{x},t) = -\frac{H(t)}{8\pi^2 \rho} \int_{S^\varepsilon} \frac{g_k g_l}{v} \dot{\delta}\left(t - \mathbf{p}_0 \cdot \mathbf{x} + \frac{1}{2}|\mathbf{x}| \sqrt{K}\left(s_1^2 + s_2^2\right)\right) ds_1 ds_2 \ . \tag{12}$$

Vectors $\mathbf{x}$ and $\mathbf{p}_0$ are parallel to the symmetry axis. The integration line in (12) is a circle and we can substitute coordinates s_1 and s_2 by polar coordinates u, φ:

$$u = \frac{1}{2}|\mathbf{x}|\sqrt{K}\left(s_1^2 + s_2^2\right) , \qquad \varphi = \mathrm{atan}\frac{s_2}{s_1} , \qquad ds_1 ds_2 = \frac{1}{\sqrt{K}}\frac{1}{|\mathbf{x}|} du d\varphi , \tag{13}$$

hence,

$$G_{kl}^{far}(\mathbf{x},t) = -\frac{H(t)}{8\pi^2 \rho v}\frac{1}{\sqrt{K}}\frac{1}{|\mathbf{x}|}\int_{S^{\varepsilon}} g_k g_l \,\dot{\delta}\left(t - \mathbf{p}_0 \cdot \mathbf{x} + u\right) du\, d\varphi . \tag{14}$$

Since the field of polarization vectors is singular at $\mathbf{p}_0$ for both degenerate waves (see Fig. 1), we cannot bring the term $g_k g_l$ in (14) in front of the integral as done in the case of the regular direction. Integrating (14) over u, we obtain

$$G_{kl}^{far}(\mathbf{x},t) = \frac{1}{8\pi^2 \rho v}\frac{1}{\sqrt{K}}\frac{\delta\left(t - \mathbf{p}_0 \cdot \mathbf{x}\right)}{|\mathbf{x}|}\int_0^{2\pi} \overline{g}_k \overline{g}_l \, d\varphi , \tag{15}$$

where

$$\overline{g}_k \overline{g}_l = \lim_{s \to 0} g_k g_l . \tag{16}$$

Specifying for SV and SH waves

$$\overline{\mathbf{g}}^{SV}\overline{\mathbf{g}}^{SV} = \begin{bmatrix} \cos^2\varphi & \sin\varphi\cos\varphi & 0 \\ \sin\varphi\cos\varphi & \sin^2\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

$$\overline{\mathbf{g}}^{SH}\overline{\mathbf{g}}^{SH} = \begin{bmatrix} \sin^2\varphi & -\sin\varphi\cos\varphi & 0 \\ -\sin\varphi\cos\varphi & \cos^2\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix} , \tag{17}$$

we get

$$\int_0^{2\pi} \overline{\mathbf{g}}^{SV}\overline{\mathbf{g}}^{SV} d\varphi = \int_0^{2\pi} \overline{\mathbf{g}}^{SH}\overline{\mathbf{g}}^{SH} d\varphi = \pi \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{18}$$

The asymptotic Green function for S waves in the kiss singularity reads

$$G_{kl}^{Sfar}(\mathbf{x},t) = G_{kl}^{SVfar}(\mathbf{x},t) + G_{kl}^{SHfar}(\mathbf{x},t) , \tag{19}$$

hence we finally obtain

$$G_{kl}^{Sfar}(\mathbf{x},t) = \frac{1}{8\pi\rho v}\left(\frac{1}{\sqrt{K^{SV}}}+\frac{1}{\sqrt{K^{SH}}}\right)\frac{\delta_{kl}-\delta_{k3}\delta_{l3}}{|\mathbf{x}|}\delta\left(t-\mathbf{p}_0\cdot\mathbf{x}\right) , \tag{20}$$

where K^{SV} and K^{SH} are the Gaussian curvatures of the *SV* and *SH* slowness sheets in the singularity, *v* is the group velocity in the singularity, and δ_{kl} is the Kronecker delta. Curvatures K^{SV} and K^{SH} are expressed in terms of elastic parameters as follows *(Vavryčuk, 1999)*:

$$K^{SV} = \frac{1}{a_{44}}\left(a_{11}-\frac{\left(a_{13}+a_{44}\right)^2}{a_{33}-a_{44}}\right)^2 , \qquad K^{SH} = \frac{a_{66}^2}{a_{44}} , \tag{21}$$

where a_{ij} are the density-normalized elastic parameters in the Voigt notation. Equations (20) and (21) are connected to the local coordinate system whose vertical axis is along the singularity. In a general coordinate system, Eq. (20) yields (*Vavryčuk, 1999, Eq. A6*)

$$\boxed{G_{kl}^{Sfar}(\mathbf{x},t) = \frac{1}{8\pi\rho v}\left(\frac{1}{\sqrt{K^{SV}}}+\frac{1}{\sqrt{K^{SH}}}\right)\frac{\delta_{kl}-n_k n_l}{|\mathbf{x}|}\delta\left(t-\frac{|\mathbf{x}|}{v}\right) ,} \tag{22}$$

where **n** denotes the direction of the phase or group velocities in the singularity.

5. KISS SINGULARITY IN GENERAL ANISOTROPY

Similarly to transverse isotropy, the *S*1 and *S*2 waves are considered to be degenerate, and the *P* wave to be non-degenerate in the singularity. Since the slowness surface can be irregular at $\mathbf{p}_0$, the Gaussian curvature may not be defined (*Shuvalov and Every, 1996*), and we can not distinguish between convex, saddle and concave shapes in the singularity. Therefore, it is convenient to introduce a generalized definition of convex, saddle and concave shapes of the surface using the normal curvature at $\mathbf{p}_0$ (for the definition of the normal curvature, see *Lipschutz, 1969, Eq. 9.14*). If the normal curvature is positive/negative for all curves crossing $\mathbf{p}_0$, the surface is convex/concave at $\mathbf{p}_0$. If the normal curvature has alternating signs at $\mathbf{p}_0$, the surface is saddle-shaped.

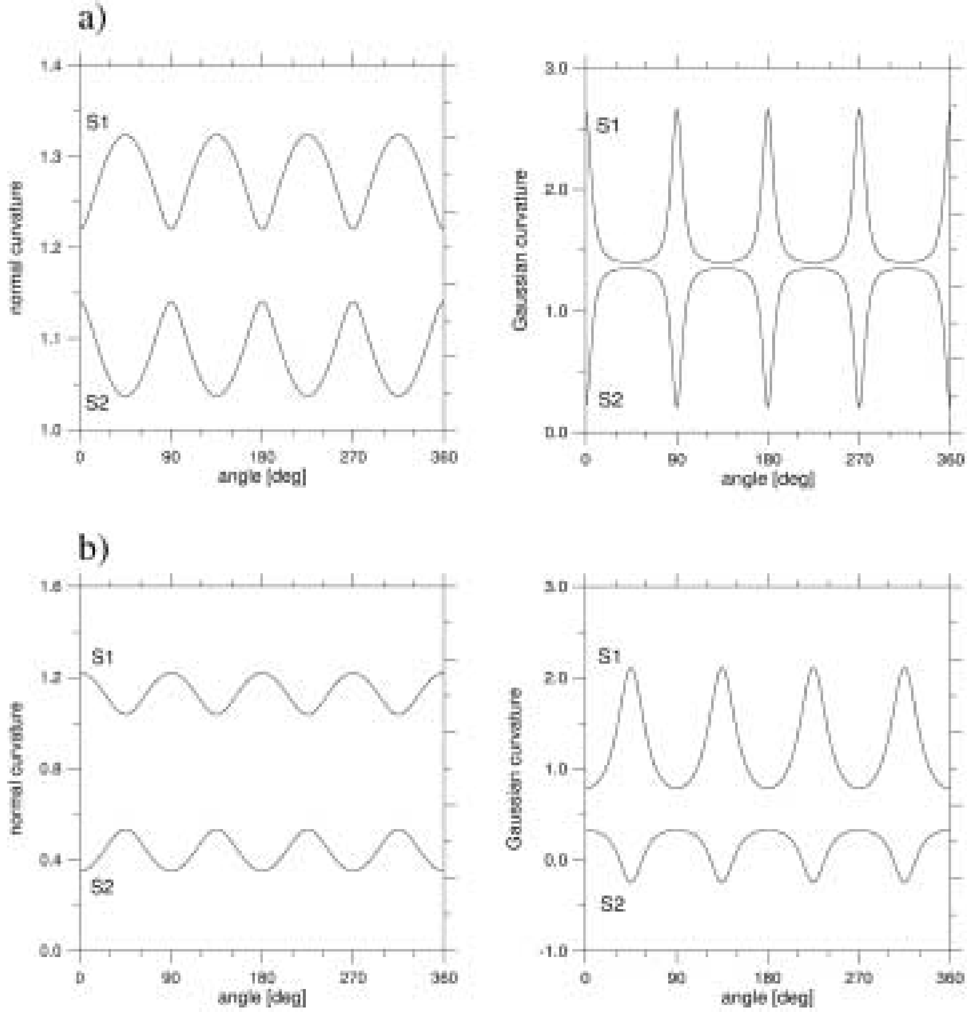

Fig. 5. Azimuthal dependence of the normal curvature in the singularity (left-hand plots) and the Gaussian curvature in the immediate vicinity of the singularity (right-hand plots) as functions of polar angle φ. The kiss singularity is along the fourfold symmetry axes in two tetragonal solids (a) and (b). For elastic parameters of the tetragonal solids, see the caption to Fig. 4.

Assuming a convex slowness surface under the above definition, the integration line in (3) is a closed curve. This curve, however, need not be an ellipse as in the case of the standard convex surface, but can be more complicated (see Fig. 4). The standard Gaussian curvature can vary along the integration line and can even attain negative values (see Fig. 5). If we substitute coordinates s_1 and s_2 by coordinates u and φ defined as:

$$u = \frac{1}{2}|\mathbf{x}|\mathbf{s}^T\bar{\mathbf{K}}\mathbf{s} = \frac{1}{2}|\mathbf{x}|\left(\bar{K}_{11}s_1^2 + 2\bar{K}_{12}s_1s_2 + \bar{K}_{12}s_2^2\right) , \qquad \varphi = \text{atan}\frac{s_2}{s_1} , \tag{23}$$

the Jacobian of the transformation becomes

$$ds_1 ds_2 = \frac{1}{|\mathbf{x}|}\frac{du d\varphi}{\bar{K}_{IJ}e_I e_J} = \frac{1}{|\mathbf{x}|}\frac{du d\varphi}{k} , \tag{24}$$

where $\mathbf{e} = (e_1, e_2)^T = (\cos\varphi, \sin\varphi)^T$, and $k(\varphi) = \bar{K}_{IJ}e_I e_J$ denotes the normal curvature in the singularity. Equation (3) can be rearranged as follows

$$G_{kl}^{far}(\mathbf{x},t) = -\frac{H(t)}{8\pi^2\rho v}\frac{1}{|\mathbf{x}|}\int_{S^\varepsilon}\frac{g_k g_l}{k}\dot{\delta}\left(t - \mathbf{p}_0\cdot\mathbf{x} + u\right) du\, d\varphi . \tag{25}$$

Integrating over u, we can write

$$G_{kl}^{far}(\mathbf{x},t) = \frac{1}{8\pi^2\rho v}\frac{\delta\left(t - \mathbf{p}_0\cdot\mathbf{x}\right)}{|\mathbf{x}|}\int_0^{2\pi}\frac{\bar{g}_k\bar{g}_l}{k}d\varphi . \tag{26}$$

Summing (26) for the $S1$ and $S2$ waves, we arrive at the final form of the asymptotic Green function in the kiss singularity in general anisotropy

$$\boxed{G_{kl}^{Sfar}(\mathbf{x},t) = \frac{1}{8\pi^2\rho v}\left(\int_0^{2\pi}\frac{\bar{g}_k^{S1}\bar{g}_l^{S2}}{k^{S1}}d\varphi + \int_0^{2\pi}\frac{\bar{g}_k^{S2}\bar{g}_l^{S2}}{k^{S2}}d\varphi\right)\frac{\delta\left(t - \mathbf{p}_0\cdot\mathbf{x}\right)}{|\mathbf{x}|} ,} \tag{27}$$

where k^{S1} and k^{S2} are the normal curvatures of the $S1$ and $S2$ slowness sheets in the singularity, and dyadics $\bar{\mathbf{g}}^{S1}\bar{\mathbf{g}}^{S1}$ and $\bar{\mathbf{g}}^{S2}\bar{\mathbf{g}}^{S2}$ are defined in (16).

To evaluate the integrals in (27) we should express polarization vectors near the singularity as a function of angle φ for both degenerate waves. Let us introduce angle Φ defining the orientation of polarization vectors $\bar{\mathbf{g}}^{S1}$ and $\bar{\mathbf{g}}^{S2}$ of the $S1$ and $S2$ waves in the immediate vicinity of the singularity:

$$\bar{\mathbf{g}}^{S1} = \mathbf{e}^{S1}\cos\Phi + \mathbf{e}^{S2}\sin\Phi , \qquad \bar{\mathbf{g}}^{S2} = -\mathbf{e}^{S1}\sin\Phi + \mathbf{e}^{S2}\cos\Phi , \tag{28}$$

where vectors $\mathbf{e}^{S1}$ and $\mathbf{e}^{S2}$ are arbitrarily chosen orthonormalized vectors in the plane perpendicular to the P-wave polarization vector $\mathbf{g}^P$. Angle $\Phi(\varphi)$ is expressed by the equation (*Shuvalov, 1998, Eq. 2.41*)

$$\cot 2\Phi(\varphi) = \frac{F_{IJ} e_I e_J}{G_{KL} e_K e_L} , \tag{29}$$

where matrices **F** and **G** are defined as (*Shuvalov, 1998, Eq. 2.16*)

$$F_{jk} = a_{ijkl}\left(e_i^{S1} e_l^{S1} - e_i^{S2} e_l^{S2}\right) + \frac{1}{\left(v^S\right)^2 - \left(v^P\right)^2}\left(q_j^{S1} q_k^{S1} - q_j^{S2} q_k^{S2}\right) ,$$

$$G_{jk} = a_{ijkl}\left(e_i^{S1} e_l^{S1} + e_i^{S2} e_l^{S2}\right) + \frac{1}{\left(v^S\right)^2 - \left(v^P\right)^2}\left(q_j^{S1} q_k^{S1} + q_j^{S2} q_k^{S2}\right) , \tag{30}$$

$$q_j^{S1} = a_{ijkl}\left(e_i^{S1} n_l + e_l^{S1} n_i\right) n_k , \quad q_j^{S2} = a_{ijkl}\left(e_i^{S2} n_l + e_l^{S2} n_i\right) n_k ,$$

where **n** denotes the direction of the slowness vector in the singularity. Note that matrices **F** and **G** also control the topological charge of the field of polarization vectors in the singularity, which can attain values 0, 1 or –1 (see *Shuvalov, 1998, Table 1*).

Substituting equations (28) – (30) into (27) we obtain the final expression for the asymptotic Green function. The complexity of the obtained integral does not allow its analytic evaluation for general anisotropy, hence the integral should be evaluated numerically. Under higher anisotropy symmetries, the integral simplifies and its analytic evaluation becomes possible.

6. KISS SINGULARITY IN TETRAGONAL AND CUBIC SYMMETRIES

Let us consider the kiss singularities along the fourfold symmetry axes in tetragonal and cubic media. The following density-normalized elastic parameters define the tetragonal symmetry (*Musgrave, 1970*):

$$a_{11} = a_{22} , \quad a_{33} , \quad a_{13} = a_{23} , \quad a_{12} , \quad a_{44} = a_{55} , \quad a_{66} , \tag{31}$$

and the cubic symmetry:

$$a_{11} = a_{22} = a_{33} , \quad a_{12} = a_{13} = a_{23} , \quad a_{44} = a_{55} = a_{66} . \tag{32}$$

Remaining elastic parameters are zero. Singularity direction $\mathbf{p}_0$ and the position vector $\mathbf{x}$ are along the vertical axis in both symmetries.

To evaluate the integrals in (27) we have to specify dyadics of polarization vectors $\bar{\mathbf{g}}^{S1}\bar{\mathbf{g}}^{S1}$ and $\bar{\mathbf{g}}^{S2}\bar{\mathbf{g}}^{S2}$ and normal curvatures k^{S1} and k^{S2} using relations (31) and (32). Defining vectors $\mathbf{e}^{S1}$ and $\mathbf{e}^{S2}$ in (28) as

$$\mathbf{e}^{S1} = (1,0,0)^T \ , \quad \mathbf{e}^{S2} = (0,1,0)^T \ , \tag{33}$$

we can express $\bar{\mathbf{g}}^{S1}\bar{\mathbf{g}}^{S1}$ and $\bar{\mathbf{g}}^{S2}\bar{\mathbf{g}}^{S2}$ as follows:

$$\bar{\mathbf{g}}^{S1}\bar{\mathbf{g}}^{S1} = \begin{bmatrix} \cos^2\Phi & \sin\Phi\cos\Phi & 0 \\ \sin\Phi\cos\Phi & \sin^2\Phi & 0 \\ 0 & 0 & 0 \end{bmatrix} ,$$

$$\bar{\mathbf{g}}^{S2}\bar{\mathbf{g}}^{S2} = \begin{bmatrix} \sin^2\Phi & -\sin\Phi\cos\Phi & 0 \\ -\sin\Phi\cos\Phi & \cos^2\Phi & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{34}$$

From (29) we obtain

$$\cos^2\Phi = \frac{1}{2}\left[1 + \frac{F_{IJ}e_Ie_J}{\sqrt{(F_{IJ}e_Ie_J)^2 + (G_{IJ}e_Ie_J)^2}}\right] ,$$

$$\sin^2\Phi = \frac{1}{2}\left[1 - \frac{F_{IJ}e_Ie_J}{\sqrt{(F_{IJ}e_Ie_J)^2 + (G_{IJ}e_Ie_J)^2}}\right] , \tag{35}$$

$$\sin\Phi\cos\Phi = \frac{1}{2}\frac{G_{IJ}e_Ie_J}{\sqrt{(F_{IJ}e_Ie_J)^2 + (G_{IJ}e_Ie_J)^2}} .$$

Matrices **F** and **G** are defined in (30) and read for tetragonal symmetry:

$$F_{11} = (a_{11}a_{33} - a_{11}a_{44} - a_{13}^2 - 2a_{13}a_{44} - a_{33}a_{66} - a_{44}^2 + a_{44}a_{66})/(a_{33} - a_{44}) \ ,$$

$$F_{12} = 0 \ , \qquad F_{22} = -F_{11} \ , \tag{36}$$

$$G_{12} = (a_{12}a_{33} - a_{12}a_{44} - a_{13}^2 - 2a_{13}a_{44} + a_{33}a_{66} - a_{44}^2 - a_{44}a_{66})/(a_{33} - a_{44}) \ ,$$

$$G_{11} = 0 \ , \qquad G_{22} = 0 \ , \tag{37}$$

and for cubic symmetry:

$$F_{11} = (a_{11}^2 - 2a_{11}a_{44} - a_{12}^2 - 2a_{12}a_{44})/(a_{11} - a_{44}) \ ,$$

$$F_{12} = 0 \ , \qquad F_{22} = -F_{11} \ , \tag{38}$$

$$G_{12} = (a_{11}a_{12} + a_{11}a_{44} - a_{12}^2 - 3a_{12}a_{44} - 2a_{44}^2)/(a_{11} - a_{44}) \ ,$$

$$G_{11} = 0 \ , \qquad G_{22} = 0 \ . \tag{39}$$

The topological charge of the field of polarization vectors in the singularity is determined as (*Shuvalov and Every, 1996, Eq. 45*)

$$n_d = \operatorname{sign}\left(F_{11}G_{12}\right) \ . \tag{40}$$

Using (36)-(39), equations (35) can be further simplified:

$$\cos^2\Phi = \frac{1}{2}\left[1 + \frac{F_{11}\cos 2\varphi}{\sqrt{F_{11}^2\cos^2 2\varphi + G_{12}^2\sin^2 2\varphi}}\right] ,$$

$$\sin^2\Phi = \frac{1}{2}\left[1 - \frac{F_{11}\cos 2\varphi}{\sqrt{F_{11}^2\cos^2 2\varphi + G_{12}^2\sin^2 2\varphi}}\right] , \tag{41}$$

$$\sin\Phi\cos\Phi = \frac{1}{2}\frac{G_{12}\cos 2\varphi}{\sqrt{F_{11}^2\cos^2 2\varphi + G_{12}^2\sin^2 2\varphi}} \ .$$

The normal curvatures k^{S1} and k^{S2} in (27) can be expressed as follows (*Shuvalov and Every, 1996, Eq. 45*):

$$k^{S1,S2}(\varphi) = \frac{1}{2}\frac{1}{\sqrt{a_{44}}}\left(f \pm \sqrt{g^2\cos^2 2\varphi + h^2\sin^2 2\varphi}\right) , \tag{42}$$

where the plus sign stands for the $S1$ wave (fast S) and the minus sign for the $S2$ wave (slow S). Quantities f, g and h are defined for tetragonal symmetry as

$$f = (a_{11}a_{33} - a_{11}a_{44} - a_{13}^2 - 2a_{13}a_{44} + a_{33}a_{66} - a_{44}^2 - a_{44}a_{66})/(a_{33} - a_{44}) \ ,$$

$$g = F_{11} \ , \qquad h = G_{12} \ , \tag{43}$$

and for cubic symmetry as

$$f = (a_{11}^2 - a_{12}^2 - 2a_{12}a_{44} - 2a_{44}^2)/(a_{11} - a_{44}) \ ,$$

$$g = F_{11} \ , \qquad h = G_{12} \ . \tag{44}$$

It follows from the stability conditions (*Helbig, 1994; Shuvalov and Every, 1996*) that $f > 0$, $f > g$ and $f > h$. Therefore, the slowness sheet of the $S1$ wave is always convex in the singularity. The slowness sheet of the $S2$ wave is convex if $f > |g|$ and $f > |h|$.

Taking into account that

$$\int_0^{2\pi} \frac{\cos 2\varphi}{\sqrt{F_{11}^2 \cos^2 2\varphi + G_{12}^2 \sin^2 2\varphi}} \ \frac{1}{f \pm \sqrt{F_{11}^2 \cos^2 2\varphi + G_{12}^2 \sin^2 2\varphi}} \ d\varphi = 0 \ , \tag{45}$$

we can simplify the integrals in (27) and obtain

$$\int_0^{2\pi} \frac{\bar{\mathbf{g}}^{S1} \bar{\mathbf{g}}^{S1}}{k^{S1}} d\varphi = \frac{\pi}{\sqrt{\bar{K}^{S1}}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} , \qquad \int_0^{2\pi} \frac{\bar{\mathbf{g}}^{S2} \bar{\mathbf{g}}^{S2}}{k^{S2}} d\varphi = \frac{\pi}{\sqrt{\bar{K}^{S2}}} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} . \tag{46}$$

Quantity $\bar{K}^{S1,S2}$ is defined as

$$\frac{1}{\sqrt{\bar{K}^{S1,S2}}} = \frac{1}{2\pi} \int_0^{2\pi} \frac{d\varphi}{k^{S1,S2}} \ , \tag{47}$$

and will be referred to as the 'generalized Gaussian curvature'. The integral in (47) can be evaluated by means of elliptic integrals, or numerically. If a surface has a regular shape, the generalized Gaussian curvature yields the value of the standard Gaussian curvature.

Finally, the asymptotic Green function in the kiss singularity in tetragonal and cubic symmetries reads

$$\boxed{G_{kl}^{Sfar}(\mathbf{x}, t) = \frac{1}{8\pi\rho v} \left(\frac{1}{\sqrt{\bar{K}^{S1}}} + \frac{1}{\sqrt{\bar{K}^{S2}}} \right) \frac{\delta_{kl} - n_k n_l}{|\mathbf{x}|} \delta\left(t - \frac{|\mathbf{x}|}{v} \right) \ ,} \tag{48}$$

where $\bar{K}^{S1}$ and $\bar{K}^{S2}$ are the generalized Gaussian curvatures of the $S1$ and $S2$ slowness sheets in the singularity, v is the group velocity, ρ is the density of the medium, $|\mathbf{x}|$ is the distance of the observation point from the source, and $\mathbf{n}$ is the direction of the phase or group velocities in the singularity. The formula is valid for the generalized convex shape of the slowness surface in the singularity.

Formula (48) derived for cubic and tetragonal symmetries is very similar to (22) derived for transverse isotropy. The only difference is that we use the generalized Gaussian curvatures instead of the standard Gaussian curvatures of the S waves in the singularity. The reason is that the shape of the slowness surface is not regular in the singularity in tetragonal or cubic symmetries, hence, the standard definition of the Gaussian curvature fails.

7. CONCLUSION

The far-field asymptotic Green function in the kiss singularity is more complex than that valid in regular directions. The complications arise due to the singularity in the field of polarization vectors and due to the irregular shape of the slowness surface. In general anisotropy, the amplitude of the Green function is expressed in the form of a 1-D integral. The integrand contains a dyadic of polarization vectors in the immediate vicinity of the singularity and an azimuthally dependent normal curvature in the singularity. The integral can be evaluated analytically for the singularity along the symmetry axis in transversely isotropic solids because the behaviour of the polarization vectors is simple, and the shape of the slowness surface is regular. The amplitude of the Green function is expressed by means of the Gaussian curvature of the slowness surface in the singularity. The kiss singularity along the fourfold symmetry axes in tetragonal and cubic media is more complicated. The shape of the slowness surface is not regular, and the Gaussian curvature is not defined in the singularity. In this case, the Gaussian curvature must be substituted by the generalized Gaussian curvature.

The derived formulae for the Green function are valid in the direction strictly along the singularity and cannot be applied to near-singularity directions. The Green function in the near-singularity directions is more complicated being affected by additional near-singularity terms.

Acknowledgments: I thank Prof. V. Borovikov, I. Pšenčík and an anonymous reviewer for their helpful reviews. This work was supported by the Grant Agency of the Czech Republic, Grant 205/00/1350 and by the Consortium project ‘Seismic waves in complex 3-D structures (SW3D)’.

Received: January 9, 2002; *Accepted: March 9, 2002*

References

Aki K. and Richards P.G., 1980. *Quantitative Seismology*, ***Vol I.***, W.H. Freeman, New York.

Alshits V.I., Sarychev A.V. and Shuvalov A.L., 1985. Classification of the degenerations and analysis of their stability in elastic wave theory in crystals. *J. Exper. Theor. Phys.*, **89**, 922-938 (in Russian).

Borovikov V.A. and Gridin D., 2001. Kiss singularities of Green’s functions of non-strictly hyperbolic equations. *Proc. R. Soc. Lond.*, **A 457**, 1059-1077.

Buchwald V.T., 1959. Elastic waves in anisotropic media. *Proc. R. Soc. Lond.*, **A 253**, 563-580.

Burridge R., 1967. The singularity on the plane lids of the wave surface of elastic media with cubic symmetry. *Q. J. Mech. Appl. Math.*, **20**, 41-56.

Burridge R., Chadwick P. and Norris A.N., 1993. Fundalmental elastodynamic solutions for anisotropic media with ellipsoidal slowness surfaces. *Proc. R. Soc. Lond.*, **A 440**, 655-681.

Crampin S. and Yedlin M., 1981. Shear-wave singularities of wave propagation in anisotropic media. *J. Geophys*., **49**, 43-46.

Červený V., 2001. *Seismic Ray Theory.* Cambridge University Press, Cambridge.

Darinskii B.M., 1994. Acoustic axes in crystals. *Crystallogr. Rep*., **39**, 697-703.

Every A.G. and Kim K.Y., 1994. Time domain dynamic response functions of elastically anisotropic solids. *J. Acoust. Soc. Amer.*, **95**, 2505-2516.

Gridin D., 2000. Far-field asymptotics of the Green tensor for a transversely isotropic solid. *Proc. R. Soc. Lond.*, **A 456**, 571-591.

Helbig K., 1994. *Foundations of Anisotropy for Exploration Seismics.* Pergamon, New York.

Lighthill M.J., 1960. Studies on magneto-hydrodynamic waves and other anisotropic wave motions. *Phil. Trans. R. Soc. London*, **A 252**, 397-430.

Lipschutz S., 1969. *Differential Geometry*. McGraw-Hill, New York.

Musgrave M.J.P., 1970. *Crystal Acoustics.* Holden-Day, London.

Payton R.G., 1983. *Elastic Wave Propagation in Transversely Isotropic Media*. Martinus Nijhoff Publishers, The Hague.

Shuvalov A.L., 1998. Topological features of the polarization fields of plane acoustic waves in anisotropic media. *Proc. R. Soc. Lond.*, **A 454**, 2911-2947.

Shuvalov A.L. and Every A.G., 1996: Curvature of acoustic slowness surface of anisotropic solids near symmetry axes. *Phys. Rev. B*, **53**, 14.906-14.916.

Shuvalov A.L. and Every A.G., 1997. Shape of the acoustic slowness surface of anisotropic solids near points of conical degeneracy. *J. Acoust. Soc. Amer.*, **101**, 2381-2382.

Vavryčuk V., 1997. Elastodynamic and elastostatic Green tensors for homogeneous weak transversely isotropic media. *Geophys. J. Int.*, **130**, 786-800.

Vavryčuk V., 1999. Properties of S waves near a kiss singularity: a comparison of exact and ray solutions. *Geophys. J. Int*., **138**, 581-589.

Vavryčuk V., 2001. Exact elastodynamic Green functions for simple types of anisotropy derived from higher-order ray theory. *Stud. Geophys. Geod.*, **45**, 67-84.

Wang C.-Y. and Achenbach J.D., 1994. Elastodynamic fundamental solutions for anisotropic solids. *Geophys. J. Int.*, **118**, 384-392.

Yeatts F.R., 1984. Elastic radiation from a point source in an anisotropic medium. *Phys. Rev. B*, **29**, 1674-1684.

APPENDIX A: MATRICES F AND G FOR GENERAL ANISOTROPY

We assume the direction of kiss singularity $\mathbf{p}_0$ along the p_3-axis and vectors $\mathbf{e}^{S1}$ and $\mathbf{e}^{S2}$ parallel to the p_1- and p_2-axis:

$$\mathbf{n} = (0,0,1)^T \ , \quad \mathbf{e}^{S1} = (1,0,0)^T \quad \text{and} \quad \mathbf{e}^{S2} = (0,1,0)^T \ . \tag{A1}$$

Inserting (A1) into (30) we obtain

$$F_{11} = (a_{11}a_{33} - a_{11}a_{44} - a_{13}^2 - 2a_{13}a_{55} - a_{33}a_{66} + a_{36}^2 + 2a_{36}a_{45} + a_{44}a_{66} + a_{45}^2 - a_{55}^2)/(a_{33}\text{-}a_{44}) \ , \tag{A2}$$

$$F_{12} = (-a_{13}a_{36} - a_{13}a_{45} + a_{16}a_{33} - a_{16}a_{44} + a_{23}a_{36} + a_{23}a_{45} - a_{26}a_{33} + a_{26}a_{44} + a_{36}a_{44} - a_{36}a_{55} + a_{44}a_{45} - a_{45}a_{55})/(a_{33} - a_{44}), \tag{A3}$$

$$F_{22} = (-a_{22}a_{33} + a_{22}a_{44} + a_{23}^2 + 2a_{23}a_{44} + a_{33}a_{66} - a_{36}^2 - 2a_{36}a_{45} + a_{44}^2 - a_{44}a_{66} - a_{45}^2)/(a_{33} - a_{44}), \tag{A4}$$

$$G_{11} = 2(-a_{13}a_{36} - a_{13}a_{45} + a_{16}a_{33} - a_{16}a_{44} - a_{36}a_{55} - a_{45}a_{55})/(a_{33} - a_{44}) \tag{A5}$$

$$G_{12} = (a_{12}a_{33} - a_{12}a_{44} - a_{13}a_{23} - a_{13}a_{44} - a_{23}a_{55} + a_{33}a_{66} - a_{36}^2 - 2a_{36}a_{45} - a_{44}a_{55} - a_{44}a_{66} - a_{45}^2)/(a_{33} - a_{44}) \ , \tag{A6}$$

$$G_{22} = 2(-a_{23}a_{36} - a_{23}a_{45} + a_{26}a_{33} - a_{26}a_{44} - a_{36}a_{44} - a_{44}a_{45})/(a_{33} - a_{44}) \ . \tag{A7}$$

Remaining elements of **F** and **G** are not specified because they are not used in the calculations.