# APPROXIMATE RELATION BETWEEN THE RAY VECTOR AND THE WAVE NORMAL IN WEAKLY ANISOTROPIC MEDIA 

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#### Abstract

Determination of the ray vector (the unit vector specifying the direction of the group velocity vector) corresponding to a given wave normal (the unit vector parallel to the phase velocity vector or slowness vector) in an arbitrary anisotropic medium can be performed using the exact formula following from the ray tracing equations. The determination of the wave normal from the ray vector is, generally, a more complicated task, which is usually solved iteratively. We present a first-order perturbation formula for the approximate determination of the ray vector from a given wave normal and vice versa. The formula is applicable to $q P$ as well as $q S$ waves in directions, in which the waves can be dealt with separately (i.e. outside singular directions of $q S$ waves). Performance of the approximate formulae is illustrated on models of transversely isotropic and orthorhombic symmetry. We show that the formula for the determination of the ray vector from the wave normal yields rather accurate results even for strong anisotropy. The formula for the determination of the wave normal from the ray vector works reasonably well in directions, in which the considered waves have convex slowness surfaces. Otherwise, it can yield, especially for stronger anisotropy, rather distorted results.


Key words: wave normal, ray vector, weak anisotropy

## 1. INTRODUCTION

It is well known that the first-order perturbations of magnitudes of phase and group velocities due to perturbation of elastic parameters are equal, see Backus (1965). However, the directions of the wave normal and of the ray vector generally differ. The determination of the ray vector corresponding to a given wave normal is an easy task (Musgrave, 1970; Červený et al., 1977; Červen'́, 2001). The determination of the wave normal from the ray vector is more complicated. It is because the wave normal and the ray vector are related by a complicated nonlinear formula (see the above references), which must be, in most cases, solved iteratively. Moreover, the formula can yield even multivalued solutions due to the triplication of $q S$-wave surfaces The problem simplifies for weakly anisotropic media. In this case, the first-order perturbation method can be applied (Jech and Pšenčik, 1989) and the formula for the ray vector can be linearized. This approach has been applied by Vavryčuk (1997, Eq. 19) for transversely isotropic media. In this paper, the approach is used for weak anisotropy of arbitrary symmetry (Pšenčilk, 1996). We expect that the derived formulae can find applications in the

[^0]estimation of elastic parameters of a homogeneous anisotropic medium from observed deviations of wave normal and ray vector. The formula can be also used in two-point ray tracing in homogeneous anisotropic media, to find the wave normal corresponding to the direction between two given points. Several applications are also mentioned by Song and Every (2000).

In Sec. 2, we summarize basic results of the first-order perturbation method for weakly anisotropic media. We use them to derive an approximate formula relating the wave normal $n_{j}$ and the ray vector $N_{j}$ and vice versa in Sec. 3. We also give approximate formulae for the group velocity vectors of all three types of waves, expressed as a function of the ray vector. In Sec. 4. accuracy of the mentioned formulae is tested on three numerical examples. Two examples represent transversely isotropic media with vertical axes of symmetry (VTI media). The third example represents an orthorhombic medium with symmetry planes coinciding with coordinate planes.

## 2. APPROXIMATE FORMULAE FOR THE PHASE VELOCITY AND POLARIZATION VECTORS

In the following, the lowercase indices attain values $1,2,3$, the uppercase indices only values 1 and 2 . Einstein summation convention is used over repeating indices.

Let us consider a weakly anisotropic medium of an arbitrary symmetry specified by the tensor of the density-normalized elastic parameters $a_{i j k l}$, and an isotropic reference medium specified by the velocity $c_{0}$, where $c_{0}=\alpha$ for the $P$ wave and $c_{0}=\beta$ for $S$ waves. The reference medium is chosen so that its elastic properties do not differ much from the weakly anisotropic medium. Let us denote by $n_{j}$ the wave normal and by $e_{j}^{(1)}$ and $e_{j}^{(2)}$ two unit vectors so that the three vectors form a mutually perpendicular triplet. In the plane perpendicular to $n_{j}$, the vectors $e_{j}^{(1)}$ and $e_{j}^{(2)}$ make an angle $\phi$ with vectors $i_{j}^{(1)}$ and $i_{j}^{(2)}$ spanning the same plane so that

$$
\begin{equation*}
e_{j}^{(1)}=i_{j}^{(1)} \cos \phi+i_{j}^{(2)} \sin \phi, \quad e_{j}^{(2)}=-i_{j}^{(1)} \sin \phi+i_{j}^{(2)} \cos \phi . \tag{1}
\end{equation*}
$$

The vectors $i_{j}^{(1)}$ and $i_{j}^{(2)}$ are mutually perpendicular unit vectors which can be defined, for example, as follows:

$$
\begin{equation*}
\vec{i}^{(1)} \equiv D^{-1}\left(n_{1} n_{3}, n_{2} n_{3}, n_{3}^{2}-1\right), \quad \vec{i}^{(2)} \equiv D^{-1}\left(-n_{2}, n_{1}, 0\right), \tag{2}
\end{equation*}
$$

where

$$
\begin{equation*}
D=\left(n_{1}^{2}+n_{2}^{2}\right)^{1 / 2}, \quad n_{1}^{2}+n_{2}^{2}+n_{3}^{2}=1 \tag{3}
\end{equation*}
$$

The vector $i_{j}^{(3)}$ is chosen so that $i_{j}^{(3)} \equiv e_{j}^{(3)} \equiv n_{j}$. For the above specification of a weakly anisotropic medium and of a wave normal, we can introduce a weak anisotropy matrix $B_{m n}^{(i)}$, see Eq. (8') in Jech and Pšenčík (1989) or Eq. (11) in Pšenčǐk and Gajewski (1998),

$$
\begin{equation*}
B_{m n}^{(i)}=\Delta a_{i j k l} n_{j} n_{l} i_{i}^{(m)} i_{k}^{(n)}=a_{i j k l} n_{j} n_{l} i_{i}^{(m)} i_{k}^{(n)}-c_{0}^{2} \delta_{m n} . \tag{4}
\end{equation*}
$$

Here $\Delta a_{i j k l}$ denotes perturbation from an isotropic reference medium, in which $c_{0}=\alpha$ for $m=n=3$ and $c_{0}=\beta$ for $m=n=1,2$. The matrix $B_{m n}^{(i)}$ results from perturbation of the Christoffel matrix, see Jech and Pšenčik (1989) and Červený (2001). It plays an important role in all formulae describing weak anisotropy. The superscript ( $i$ ) indicates that the matrix $B_{m n}^{(i)}$ is calculated using the vectors $i_{j}^{(m)}$.

For a fixed wave normal, the perturbation theory yields the following formulae for the phase velocity and polarization vectors. The phase velocity in a weakly anisotropic medium is given by (see Eqs. (8) and (18) of Jech and Pšenčík, 1989)

$$
\begin{equation*}
c_{3}=c_{0}+\Delta c_{3} \cong \alpha+\frac{1}{2 \alpha} B_{33}^{(i)} \tag{5a}
\end{equation*}
$$

for the $q P$ wave and by

$$
\begin{equation*}
c_{1}=c_{0}+\Delta c_{1} \cong \beta+\frac{1}{2 \beta} B_{11}^{(e)}, \quad c_{2}=c_{0}+\Delta c_{2} \cong \beta+\frac{1}{2 \beta} B_{22}^{(e)} \tag{5b}
\end{equation*}
$$

for $q S$ waves. The symbol $\Delta c_{3}$ denotes perturbation of the phase velocity for $q P$ wave and $\Delta c_{1}$ and $\Delta c_{2}$ for $q S$ waves. The symbol $B_{m n}^{(e)}$ denotes an element of the weak anisotropy matrix calculated using vectors $e_{j}^{(I)}$, see below.

The polarization vector of the $q P$ wave is given by (see Eqs. (11) and (23) of Jech and Pšenčík, 1989)

$$
\begin{equation*}
g_{j}^{(3)}=n_{j}+\Delta g_{j}^{(3)} \cong n_{j}+\frac{B_{13}^{(i)} i_{j}^{(1)}+B_{23}^{(i)} i_{j}^{(2)}}{\alpha^{2}-\beta^{2}} . \tag{6a}
\end{equation*}
$$

The polarization vectors of $q S$ waves are given by

$$
\begin{align*}
& g_{j}^{(1)}=e_{j}^{(1)}+\Delta g_{j}^{(1)} \cong e_{j}^{(1)}+\frac{B_{13}^{(e)}}{\beta^{2}-\alpha^{2}}\left(n_{j}+\frac{B_{23}^{(e)} e_{j}^{(2)}}{B_{11}^{(e)}-B_{22}^{(e)}}\right), \\
& g_{j}^{(2)}=e_{j}^{(2)}+\Delta g_{j}^{(2)} \cong e_{j}^{(2)}+\frac{B_{23}^{(e)}}{\beta^{2}-\alpha^{2}}\left(n_{j}-\frac{B_{13}^{(e)} e_{j}^{(1)}}{B_{11}^{(e)}-B_{22}^{(e)}}\right) . \tag{6b}
\end{align*}
$$

The matrix $B_{m n}^{(e)}$ is a symmetric matrix whose elements are expressed in terms of elements of the matrix $B_{m n}^{(i)}$ and especially chosen angle $\phi$ for which $B_{12}^{(e)}=0$. This choice of angle $\phi$ yields the vectors $e_{j}^{(I)}$, which can be used as zero-order approximation of the
polarization vectors of $q S$ waves in the isotropic reference medium. The elements of the matrices $B_{m n}^{(e)}$ and $B_{m n}^{(i)}$ are related in the following way:

$$
\begin{gather*}
B_{11}^{(e)}=B_{11}^{(i)} \cos ^{2} \phi+2 B_{12}^{(i)} \cos \phi \sin \phi+B_{22}^{(i)} \sin ^{2} \phi, \\
B_{22}^{(e)}=B_{11}^{(i)} \sin ^{2} \phi-2 B_{12}^{(i)} \cos \phi \sin \phi+B_{22}^{(i)} \cos ^{2} \phi, \\
B_{12}^{(e)}=\left(B_{22}^{(i)}-B_{11}^{(i)}\right) \cos \phi \sin \phi+B_{12}^{(i)}\left(\cos ^{2} \phi-\sin ^{2} \phi\right), \\
B_{33}^{(e)}=B_{33}^{(i)}, \quad B_{13}^{(e)}=B_{13}^{(i)} \cos \phi+B_{23}^{(i)} \sin \phi, \quad B_{23}^{(e)}=-B_{13}^{(i)} \sin \phi+B_{23}^{(i)} \cos \phi . \tag{7}
\end{gather*}
$$

Let us note that equations (6b) fail when $B_{11}^{(e)}$ is close to $B_{22}^{(e)}$. This happens in the vicinity of the $q S$-wave singularities. From the equation $B_{12}^{(e)}=0$, we can find the formula for the determination of the angle $\phi$,

$$
\begin{equation*}
\tan 2 \phi=\frac{2 B_{12}^{(i)}}{B_{11}^{(i)}-B_{22}^{(i)}} \tag{8}
\end{equation*}
$$

Alternatively, the matrix $B_{m n}^{(e)}$, can be expressed in the following way analogous to (4)

$$
\begin{equation*}
B_{m n}^{(e)}=a_{i j k l} n_{j} n_{l} e_{i}^{(m)} e_{k}^{(n)}-c_{0}^{2} \delta_{m n} \tag{9}
\end{equation*}
$$

The vectors $e_{j}^{(I)}$ in Eq. (9) are given by (1) with $\phi$ specified in Eq. (8). From (5) and (6), we can see that the matrix $B_{m n}^{(e)}$ appears only in the expressions for $q S$ waves.

Let us mention that approximate values of phase velocities in the examples shown later were calculated by taking square roots of the following first-order formulae for the square of the $q P$-wave phase velocity,

$$
\begin{equation*}
c_{3}^{2} \cong \alpha^{2}+B_{33}^{(i)} \tag{10a}
\end{equation*}
$$

and for the squares of the $q S$-wave phase velocities

$$
\begin{equation*}
c_{1}^{2} \cong \beta^{2}+B_{11}^{(e)}, \quad c_{2}^{2} \cong \beta^{2}+B_{22}^{(e)} . \tag{10b}
\end{equation*}
$$

Eqs. (10) follow straightforwardly from (5) by neglecting second-order terms. Using Eq. (4) in Eq. (10a) we get for $q P$ waves

$$
\begin{equation*}
c_{3}^{2} \sim a_{i j k l} n_{i} n_{j} n_{k} n_{l} \tag{11a}
\end{equation*}
$$

see Eq. (14) in Pšenčík and Gajewski (1998). Using Eq. (9) in Eqs. (10b), we get for $q S$ waves

$$
\begin{equation*}
c_{1}^{2} \cong a_{i j k l} n_{j} n_{l} e_{i}^{(1)} e_{k}^{(1)}, \quad c_{2}^{2} \cong a_{i j k l} n_{j} n_{l} e_{i}^{(2)} e_{k}^{(2)} \tag{11b}
\end{equation*}
$$

For similar expressions, see Farra (2001). We can see that the expressions for the square of the phase velocity (11) are independent of the choice of the parameters of the reference medium.

## 3. RELATION BETWEEN WAVE NORMAL AND RAY VECTOR

The approximate formulae for the phase velocities and polarization vectors presented in the previous section follow from formulae derived by Jech and Pšenčik (1989). In this section, we discuss a different topic, namely approximate relation of the ray vector and the wave normal, and approximate formulae for the group velocity. Let us start from the formula for the group velocity vector $v_{j}$ in an anisotropic medium, see e.g. Musgrave (1970), Červený et al. (1977), Červený (2001):

$$
\begin{equation*}
v_{j}=v N_{j}=c^{-1} a_{i j k l} n_{l} g_{i} g_{k} \tag{12}
\end{equation*}
$$

For simplicity, the superscripts denoting the type of the considered wave are omitted. In Eq. (12), $v$ and $c$ are the group and phase velocities and $g_{j}$ denotes the polarization vector of a considered wave. Generally, the group- and phase-velocity vectors differ in anisotropic media by their magnitudes $v$ and $c$ as well as directions $N_{j}$ and $n_{j}$. The determination of the ray vector $N_{j}$ from the wave normal $n_{j}$ can be performed with the use of exact relation (12). For approximate determination of $n_{j}$ from $N_{j}$, we use the first-order perturbation method. We seek the ray vector $N_{j}$ as a perturbation of the wave normal $n_{j}$,

$$
\begin{equation*}
N_{j}=n_{j}+\Delta N_{j}=n_{j}+A_{1} e_{j}^{(1)}+A_{2} e_{j}^{(2)}, \tag{13a}
\end{equation*}
$$

where $A_{I}(I=1,2)$ are coefficients to be determined. The group velocity $v$ can be expressed as

$$
\begin{equation*}
v=c_{0}+\Delta v, \tag{13b}
\end{equation*}
$$

where $\Delta v$ is the perturbation of the group velocity
Using Eqs. (13) and neglecting second-order perturbations, the group velocity vector $v_{j}$ can be expressed as a sum of the component parallel to the wave normal and a component perpendicular to it:

$$
\begin{equation*}
v_{j}=\left(c_{0}+\Delta v\right) n_{j}+c_{0}\left(A_{1} e_{j}^{(1)}+A_{2} e_{j}^{(2)}\right) \tag{14}
\end{equation*}
$$

Similarly, we can expand the RHS of Eq. (12) for wave normal kept fixed. In such a way, we get an alternative expression for the group velocity vector $v_{j}$ :

$$
\begin{align*}
v_{j}= & c_{0} n_{j}-\Delta c n_{j}+c_{0}^{-1}\left[a_{i j k l} n_{l} e_{i} e_{k}\right. \\
& \left.+\left(\alpha^{2}-\beta^{2}\right)\left(\left(n_{k} e_{k}\right) \Delta g_{j}-\left(n_{k} e_{k}\right) e_{j}+\left(n_{k} \Delta g_{k}\right) e_{j}\right)-\beta^{2} n_{j}\right] . \tag{15}
\end{align*}
$$

Comparison of projections of group velocity vector $v_{j}$ given by (14) and (15) onto the wave normal yields

$$
\begin{equation*}
\Delta v=\Delta c \tag{16}
\end{equation*}
$$

This is a well-known identity (Backus, 1965), indicating that the phase and group velocities have equal magnitudes in the first-order approximation of the perturbation theory.

Comparing projections of the group velocity vector $v_{j}$ given by (14) and (15) onto the vector $e_{j}^{(I)}$ yields sought coefficients $A_{I}$ :

$$
\begin{align*}
A_{I}= & c_{0}^{-2}\left[a_{i j k l} n_{l} e_{i} e_{k} e_{j}^{(I)}\right. \\
& \left.+\left(\alpha^{2}-\beta^{2}\right)\left(\left(n_{k} e_{k}\right)\left(\Delta g_{j} e_{j}^{(I)}\right)-\left(n_{k} e_{k}\right)\left(e_{j} e_{j}^{(I)}\right)+\left(n_{k} \Delta g_{k}\right)\left(e_{j} e_{j}^{(I)}\right)\right)\right] \tag{17}
\end{align*}
$$

Using (13a) and (17) we can write an approximate expression relating the ray vector $N_{j}$ and the wave normal $n_{j}$

$$
\begin{equation*}
N_{j}=n_{j}+\Delta N_{j}\left(n_{k}\right), \tag{18}
\end{equation*}
$$

where the term $\Delta N_{j}$ reads for the $q P$ wave:

$$
\begin{equation*}
\Delta N_{j}^{(3)}=2 \alpha^{-2}\left(B_{13}^{(i)} i_{j}^{(1)}+B_{23}^{(i)} i_{j}^{(2)}\right) \tag{19a}
\end{equation*}
$$

and for $q S 1$ and $q S 2$ waves:

$$
\begin{align*}
& \Delta N_{j}^{(1)}=\beta^{-2}\left(\Delta a_{i j k l} n_{l} e_{i}^{(1)} e_{k}^{(1)}-B_{11}^{(e)} n_{j}-B_{13}^{(e)} e_{j}^{(1)}\right), \\
& \Delta N_{j}^{(2)}=\beta^{-2}\left(\Delta a_{i j k l} n_{l} e_{i}^{(2)} e_{k}^{(2)}-B_{22}^{(e)} n_{j}-B_{23}^{(e)} e_{j}^{(2)}\right) . \tag{19b}
\end{align*}
$$

Since $A_{I}$ is a first-order quantity, Eq. (13a) implies that everywhere, where a first-order quantity is multiplied by $N_{j}$ or $n_{j}$, these two vectors are interchangeable. This means that, within the first-order approximation, with respect to $\Delta a_{i j k l}$, there is no difference if the perturbation $\Delta N_{j}$ is expressed with respect to $n_{j}$ or $N_{j}$. This makes possible to rewrite Eq. (18) within the same first-order approximation, into the form

$$
\begin{equation*}
n_{j}=N_{j}-\Delta N_{j}\left(N_{k}\right) \tag{20}
\end{equation*}
$$

In (20), $\Delta N_{j}\left(N_{k}\right)$ is determined from (19) with $n_{j}$ substituted by $N_{j}$. Eq. (20) should work in directions, in which the considered wave has a convex slowness surface. It cannot work, however, when the slowness surface is concave or hyperbolic (this indicates triplication of the group velocity surface and three values of the function $n_{j}=n_{j}\left(N_{k}\right)$ for one $N_{k}$ ). Since Eq. (20) can yield only a single value of $n_{j}$ for a value of $N_{k}$, the equation cannot, in principle, describe properly the exact behaviour of the function $n_{j}=n_{j}\left(N_{k}\right)$. In addition, we should expect a worse performance of Eq. (20) than of Eq. (18), caused by the strong nonlinearity of the term $\Delta N_{j}$ with respect to the wave normal $n_{j}$. Eqs. (18) and
(20) with (19) represent sought approximate formulae for the determination of the ray vector from wave normal and vice versa.

From (14), (16), (18) and (19), we can also derive approximate formulae for the group velocity vector in a weakly anisotropic medium. The formulae can be expressed both in terms of the wave normal $n_{j}$ and the ray vector $N_{j}$. For $q P$ waves, the group velocity vector has the form

$$
\begin{equation*}
v_{j}^{(3)}=\left(\alpha+\Delta c_{3}\left(n_{k}\right)\right) n_{j}+\alpha \Delta N_{j}^{(3)}\left(n_{k}\right)=\left(\alpha+\Delta c_{3}\left(N_{k}\right)\right) N_{j}^{(3)} . \tag{21a}
\end{equation*}
$$

For $q S 1$ and $q S 2$ waves, the group velocity vectors are

$$
\begin{align*}
& v_{j}^{(1)}=\left(\beta+\Delta c_{1}\left(n_{k}\right)\right) n_{j}+\beta \Delta N_{j}^{(1)}\left(n_{k}\right)=\left(\beta+\Delta c_{1}\left(N_{k}\right)\right) N_{j}^{(1)}, \\
& v_{j}^{(2)}=\left(\beta+\Delta c_{2}\left(n_{k}\right)\right) n_{j}+\beta \Delta N_{j}^{(2)}\left(n_{k}\right)=\left(\beta+\Delta c_{2}\left(N_{k}\right)\right) N_{j}^{(2)} . \tag{21b}
\end{align*}
$$

For $\Delta c_{k}$ and $\Delta N_{j}^{(k)}$ see (5) and (19), respectively.
Let us mention an interesting phenomenon. If we compare Eq. (6a) and Eq. (21a) and take into account Eq. (19a), we can see that the $q P$-wave polarization vector, $q P$-wave group velocity vector and the wave normal are coplanar in weakly anisotropic media. This is an extension of the observation made by Crampin (1981) for general anisotropy in symmetry planes. As Crampin (1981), we can also observe that the deviation of the group velocity and polarization vectors from the wave normal is in the same direction. All three vectors become parallel in the longitudinal directions (Helbig, 1994). In this case, the elements $B_{I 3}^{(i)}$ of the weak anisotropy matrix vanish.

## 4. NUMERICAL EXAMPLES

We illustrate the accuracy of the derived formulae on three models of anisotropic media. Model A and Model B are transversely isotropic with vertical axes of symmetry (VTI). Model A is the Shearer and Chapman (1989) Model 1 (thin water-filled cracks), Model B is Model 4 of the same authors (thin water-filled cracks - extremely anisotropic). The Model C is orthorhombic, adopted from Farra (2001). Anisotropy (calculated as $\left.2\left(c_{\max }-c_{\min }\right) /\left(c_{\max }+c_{\min }\right) \times 100 \%\right)$ of Model A is about $3.5 \%$ for $q P$ wave and about $11.2 \%$ for $q S$ waves. Anisotropy of Model B is about $9 \%$ for $q P$ wave, $29 \%$ and $30 \%$ for $q S$ waves. We can see that anisotropy of $q S$ waves in Model B is rather strong. Anisotropy of Model C is $14 \%$ for $q P$ waves and $7 \%$ and $5 \%$ for $q S$ waves. Reference velocities used are $\alpha=4.41$ and $\beta=2.42 \mathrm{~km} / \mathrm{s}$ in Model A, $\alpha=4.30$ and $\beta=2.28 \mathrm{~km} / \mathrm{s}$ in Model B, and $\alpha=3.17$ and $\beta=2.00 \mathrm{~km} / \mathrm{s}$ in Model C.


Fig. 1. Test of approximate formulae (18) (top) and (20) (bottom) for $q P$ wave in Model B. Approximate curves (dotted lines) are compared with exact curves (solid lines). The wave normal and the ray vector are specified by their angles (in degrees) with the axis of symmetry.


Fig. 2. Test of approximate formulae (18) (left column) and (20) (right column) for $q S$ waves in Model A (top) and Model B (bottom). qSV wave - blue, SH wave - red. Approximate curves (dotted lines) are compared with exact curves (solid lines). The wave normal and the ray vector are specified by their angles (in degrees) with the axis of symmetry.

### 4.1 VTI Symmetry

Model A is characterized by the density-normalized elastic parameters $A_{i j}$, in $(\mathrm{km} / \mathrm{s})^{2}$, with values: $A_{11}=A_{22}=20.22, A_{33}=20.04, A_{12}=7.46, A_{13}=A_{23}=7.41$, $A_{44}=A_{55}=5.10, A_{66}=6.38$. Model B is characterized by the density-normalized elastic parameters $A_{i j}$, in $(\mathrm{km} / \mathrm{s})^{2}$, with values: $A_{11}=A_{22}=20.16, A_{33}=19.63, A_{12}=7.40$, $A_{13}=A_{23}=7.26, A_{44}=A_{55}=3.48, A_{66}=6.38$.

The accuracy of approximate formulae (18) and (20) for the determination of the ray vector $N_{j}$ from a given wave normal $n_{j}$ and vice versa is generally rather high for the $q P$ wave as can be seen from Figure 1. The upper plot shows results obtained with approximate formula (18), the bottom plot with formula (20) for Model B. The wave normal and the ray vector are specified by their angles (in degrees) with the axis of
symmetry. The comparison of approximate curves shown by dotted lines with exact curves shown by solid lines indicates quite a high accuracy of the approximate formulae.

The upper plots in Figure 2 show a comparison of the exact (solid line) and approximate (dotted line) curves for Model A. The bottom plots show the same for Model B. The left-hand plots show $N_{j}=N_{j}\left(n_{k}\right)$, see Eq. (18), the right-hand plots show $n_{j}=n_{j}\left(N_{k}\right)$, see Eq. (20). The angles of $0^{\circ}$ correspond to the wave normal or ray vector along the symmetry axis, the angles of $90^{\circ}$ correspond to the wave normal or ray vector perpendicular to the symmetry axis. For an isotropic medium the curves would be straight lines. Deviations from such lines indicate anisotropy. The exact and approximate curves $N_{j}=N_{j}\left(n_{k}\right)$ for Model A match each other very well for both $q S$ waves The approximate formula $n_{j}=n_{j}\left(N_{k}\right)$ for the $S H$ wave in Model A is of a similar accuracy. The accuracy is, however, rather low for the $q S V$ wave. The approximate curve only indicates the basic trend of the exact curve. This effect is considerably more pronounced in the case of the $q S V$ wave in Model B. The approximate formula fails because the single-valued approximate curve cannot fit the multi-valued exact curve. This is a consequence of the strong nonlinearity of the term $\Delta N_{j}$ with respect to the wave normal $n_{j}$. The approximate curves $N_{j}=N_{j}\left(n_{k}\right)$ and $n_{j}=n_{j}\left(N_{k}\right)$ for the $S H$ wave for Model B show slightly greater deviations from the exact ones (anisotropy of Model B is rather strong) than in the Model A. The approximate curve $N_{j}=N_{j}\left(n_{k}\right)$ for the $q S V$ wave has comparable accuracy. In this case, some ray vector directions correspond to three different phase normal directions. This is an indication of the triplication of the corresponding wavefront.

The accuracy of the formula (18) for Model A is illustrated in Figure 3. The upper plot shows exact angular deviation of the ray vector of $q S$ waves from the wave normal as a function of the direction of the wave normal. The bottom plot shows the angular errors of the approximate formula (18): deviations of the approximate ray vector from the exact one in degrees. We can see in the upper plot that the deviation of the ray vector from the wave normal is always to one side and can reach $12^{\circ}$ for the $q S V$ wave and $6^{\circ}$ for the $S H$ wave at maximum. The maximum errors of the approximate formula (18) are about $0.45^{\circ}$ for the $q S V$ wave and nearly $0.5^{\circ}$ for the $S H$ wave. The above curves have similar forms also for Model B, only values are different. The maximum deviation of the ray vector from the wave normal can reach $30^{\circ}$ for the $q S V$ wave and $20^{\circ}$ for the $S H$ wave. The maximum errors of the approximate formula (18) are about $3.5^{\circ}$ for both $q S$ waves in Model B.

Figure 4 shows sections of the group velocity surfaces of $q S V$ and $S H$ waves as a function of the direction of the ray vector, see (21b). The ray vectors $N_{j}$ in (21b) are determined from (18) and (19b) for regularly specified wave normals $n_{j}$. The upper plot corresponds to Model A, the bottom plot to Model B. We can see that (21b) yields, generally, less accurate results than previous formulae. It is due to the combination of two approximations contained in (21b): the approximation of the direction and the approximation of the value of the group velocity. Due to Eq. (16), the group velocity is equal to the phase velocity in the first-order perturbation approximation. An interesting phenomenon can be observed on the curve corresponding to the section of the $q S V$ wave group-velocity surface in Model B. It shows that even the approximate formula can very roughly describe triplication of the group velocity surface.


Fig. 3. Exact angular differences of the ray vector and the wave normal (top), and error plot of angular deviations of the approximate (Eq. (18)) and exact ray vectors (bottom) for $q S$ waves in Model A. $q S V$ wave - blue, $S H$ wave - red. The wave normal is specified by its angle (in degrees) with the axis of symmetry.


Fig. 4. Test of approximate formulae (21b) for the group velocity of $q S$ waves in Model A (top) and Model B (bottom). $q S V$ wave - blue, $S H$ wave - red. The approximate curves (dotted lines) are compared with exact curves (solid lines). Ray vector is specified by its angle (in degrees) with the axis of symmetry.

### 4.2 Orthorhombic Symmetry

Model C is characterized by the density-normalized elastic parameters $A_{i j}$, in $(\mathrm{km} / \mathrm{s})^{2}$, with values: $A_{11}=10.8, A_{22}=11.3, A_{33}=8.5, A_{12}=2.2, A_{13}=1.9, A_{23}=1.7, A_{44}=3.6$, $A_{55}=3.9, A_{66}=4.3$.

Figure 5 illustrates accuracy of the approximate formula (18) for the $q S 1$ wave (faster of the $q S$ waves). The upper picture shows equal area plot of exact angular deviations (in degrees) of the ray vector and the wave normal. The bottom picture shows equal area plot of angular errors of the approximate formula (18). Both plots are parameterized by the wave normal. The maximum errors of the formula (18) are comparable with maximum exact differences between the wave normal and the ray vector. It is, however, necessary to emphasize that the maximum errors are concentrated to very narrow strips close to $45^{\circ}$ inclination of the wave normal. In these directions the $q S$ waves propagate with nearly the same phase velocities, i.e., the mentioned directions are singular directions, in which the studied formulae cannot, in principle, work properly. For the remaining directions, the errors are less than $1^{\circ}$. For the $q P$ wave, the formula (18) yields errors of the same order everywhere. The errors are slightly larger for the $q S 2$ wave. For all the waves, the approximate formulae work very well in the horizontal and vertical directions, which correspond to longitudinal directions.

## 5. CONCLUSIONS

Approximate formulae relating the ray vector and the wave normal were presented. The formulae are applicable to $q P$ wave generally and to $q S$ waves outside singular directions. The formulae can be also used for the approximate evaluation of the group velocity vector from the ray vector.

The accuracy of the formulae was tested on three examples of models of anisotropic media. The results show that the approximate formulae $N_{j}=N_{j}\left(n_{k}\right)$ and $n_{j}=n_{j}\left(N_{k}\right)$ work better for the $q P$ wave than for the $q S$ waves. For the $q S$ waves, the above formulae fail in singular directions and their vicinities. Distorted results are also obtained from the formula $n_{j}=n_{j}\left(N_{k}\right)$ in directions of the triplication of the group velocity surface. The approximate formula $N_{j}=N_{j}\left(n_{k}\right)$ works rather well even in these directions. We have shown that the approximate formula for the group velocity as a function of the ray vector can even roughly describe a triplication.

The performed study shows that the approximate formulae relating the wave normal and the ray vector can be used quite safely for $q P$ waves. For $q S$ waves they must be used with a great care, especially when the wave normal is sought from a given ray vector.

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qS1 wave
Fig. 5. Equal area plots of angular differences of the exact ray vector and the wave normal (top) and of angular deviations of the approximate (Eq. (18)) and exact ray vectors (bottom) for $q S 1$ wave in Model C. Parameterization is in terms of wave normal, horizontal and vertical component of the wave normal along horizontal and vertical axis, respectively.

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