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SH-wave Green tensor for homogeneous transversely isotropic media by higher-order approximations in asymptotic ray theory

Václav Vavryčuk *, Kiyoshi Yomogida

Department of Earth and Planetary Systems Science, Faculty of Science, Hiroshima University, 1-3-1 Kagami-yama, Higashi-Hiroshima 739, Japan

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Abstract

Tensor equations of the ray theory for homogeneous anisotropic elastic media are presented. For a point source, an explicit solution of the transport equation is obtained, thus the additional as well as principal components of the ray amplitudes for higher-order ray approximations are expressed only by differential operators of lower-order terms. Possibility of analytical calculation of higher-order approximations is exemplified for SH waves in a transversely isotropic medium. The ray series of the SH-wave Green tensor for the transversely isotropic medium involves only two non-zero terms and the complete ray solution coincides with an exact solution obtained by other complicated procedures.

1. Introduction

The ray equations for elastic waves in anisotropic media were first developed by Babich [1] and Červený [9]. Consequently, they were studied and applied by other authors (e.g., Červený et al. [10]; Gajewski and Pšenčík [11,12]; Norris [20]; Kendall and Thomson [14]; Ben-Menahem et al. [4]) who paid attention in particular to numerical aspects and the development of effective computer codes for modelling wavefields in the so-called zeroth-order ray approximation. Apparently, the consideration of higher-order ray approximations can lead us to the further extension of ray theoretical applications. Including the higher approximations can extend the validity of the ray method to some areas, where the conventional ray method fails (for a review see Babich and Kiselev [2]). Let us mention as an example modelling of near-field waves in anisotropic media, which is performed so far by computationally demanding algorithms (e.g., Mallick and Frazer [17]; Mandal and Toksöz [18]; Tsingas et al. [24]; Tsvankin and Chesnokov [25], Carcione et al. [8]; Vavryčuk [27]). In our paper we present an analytical approach to the calculation of higher-order ray approximations constraint to homogeneous anisotropic media. Although an exact formula for the Green tensor for such media is not simple (Buchwald [5]; Lighthill [16]; Burridge [6]; Musgrave [19]; Tverdokhlebov and Rose [26]), its approximation for the far-field waves is written in an explicit analytical form (e.g., Burridge [6]; Ben-Menahem et al. [4]; Kendall et al. [15]). This approximation represents, in fact, the zeroth-order term of the ray theory and it can serve for the recurrent computation of each higher-order term of the ray series that physically means the near-field wave. For analytical calculations of the higher-order

* Corresponding author. On leave from: Geophysical Institute, Czech Academy of Sciences, Boční II/1401, 141 31 Praha 4, Czech Republic. E-mail vasek@seis.ig.cas.cz.

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approximations we use a transversely isotropic medium, which has already been studied by many authors (e.g., Payton [21]; Carcione et al. [7]; Kazi-Aoual et al. [13]; Ben-Menahem and Sena [3]; Sakai and Kawasaki [22]) and it is widely applied in seismological applications. By calculating the Green tensor for SH-waves in transversely isotropic media we show that the higher-order ray approximations can give useful results and that the ray theory can even produce an *exact* solution.

2. Higher-order approximations of the ray theory

2.1. Basic equations of the ray theory

The elastodynamic Green tensor for homogeneous anisotropic media satisfies the equation

$$\rho \ddot{G}_{in} - c_{ijkl} G_{kn,lj} = \delta_{in} \delta(x) \delta(t) , \tag{1}$$

where G_{in} is the symmetric tensor of the second rank, ρ is the density, c_{ijkl} is the elasticity tensor, δ_{in} is the Kronecker delta, and $\delta(t)$ is the Dirac delta function. Einstein summation convention is applied, where repeated indices mean summation. We seek a solution in a form of the ray series [1,10]:

$$G_{in}(x_j, t) = \sum_{K=0}^{\infty} U_{in}^{(K)}(x_j) f^{(K)}(t - \tau(x_j)) ,$$

$$\text{where} \quad \frac{\mathrm{d}}{\mathrm{d}t} f^{(K)}(t) = f^{(K-1)}(t) . \tag{2}$$

K denotes the order of the approximation, $U_{in}^{(K)}(x_j)$ is the ray amplitude tensor and $\tau(x_j)$ is the traveltime. Inserting formula (2) into Eq. (1) leads to a recurrent system of equations for the amplitude tensors $U_{in}^{(K)}$:

$$N_{in}(U_{kn}^{(K)}) - M_{in}(U_{kn}^{(K-1)}) + L_{in}(U_{kn}^{(K-2)}) = 0 , \tag{3}$$

called the *basic equations of the ray theory*. Differential tensor operators N_{jn}, M_{jn} and L_{jn} are defined for homogeneous anisotropic media as follows

$$N_{jn}(U_{kn}^{(K)}) = \Gamma_{jk} U_{kn}^{(K)} - U_{jn}^{(K)} , \quad M_{jn}(U_{kn}^{(K)}) = a_{ijkl}(p_i U_{kn,l}^{(K)} + p_l U_{kn,i}^{(K)} + p_{i,l} U_{kn}^{(K)}) ,$$

$$L_{jn}(U_{kn}^{(K)}) = a_{ijkl} U_{kn,il}^{(K)} , \quad a_{ijkl} = \frac{c_{ijkl}}{\rho} , \quad \Gamma_{jk} = a_{ijkl} p_i p_l , \; p_i = \tau_{,i} . \tag{4}$$

a_{ijkl} is the normalized elasticity tensor, Γ_{jk} is the Christoffel tensor and p_i is the slowness vector. The ray amplitude $U_{kn}^{(K)}$ for $K < 0$ equals zero.

The only differences between our equations and those presented by Babich [1], Červený [9] or Červený et al. [10] lie in using homogeneous media instead of general inhomogeneous media, and in using tensors instead of vectors. Since the generalization of the vector notation to our tensor notation is straightforward, we do not give a detailed derivation.

2.2. Principal and additional components

In this section, we will present formulae for higher-order ray approximations of the P-wave. For S1- or S2-waves, the equations can be rewritten in an analogous way except for shear-wave singular points where the eigenvalues of the Christoffel tensor corresponding to the two shear waves are identical.

For calculation of higher-order ray approximations, it is convenient to introduce so-called *additional* and *principal components* $U_{mn}^{P(K)\perp}$ and $U_{mn}^{P(K)\parallel}$ of the ray amplitude $U_{mn}^{P(K)}$ ($K \geqslant 0$)

$$U_{mn}^{\mathrm{P}(K)} = U_{mn}^{\mathrm{P}(K)\perp} + U_{mn}^{\mathrm{P}(K)\parallel} . \tag{5}$$

Calculation of each higher-order term of the ray series is performed in two steps: first, the additional component $U_{mn}^{\mathrm{P}(K)\perp}$ is calculated by the differentiation of the lower-order terms, and second, the principal component $U_{mn}^{\mathrm{P}(K)\parallel}$ is calculated by solving an ordinary differential equation called the *transport equation.*

The additional component for the P-wave is calculated by the following formula corresponding to the vector formula (22) of Červený [9],

$$U_{mn}^{\mathrm{P}(K)\perp} = \{M_{in}(U_{kn}^{\mathrm{P}(K-1)}) - L_{in}(U_{kn}^{\mathrm{P}(K-2)})\}\left\{\frac{g_i^{\mathrm{S1}} g_m^{\mathrm{S1}}}{G^{\mathrm{S1}} - G^{\mathrm{P}}} + \frac{g_i^{\mathrm{S2}} g_m^{\mathrm{S2}}}{G^{\mathrm{S2}} - G^{\mathrm{P}}}\right\}, \tag{6}$$

where G^{P}, G^{S1} and G^{S2} are the eigenvalues and g_j^{P}, g_j^{S1} and g_j^{S2} are the eigenvectors of the Christoffel tensor. Superscripts P, S1 and S2 indicate the wave types. The eigenvalue G^{P} equals 1. Obviously, the additional component of the zeroth-order term $U_{mn}^{\mathrm{P}(0)\perp}$ equals zero.

The principal component $U_{mn}^{\mathrm{P}(K)\parallel}$ is calculated by solving the transport equation (Červený [9], formula (28), which can be specified for homogeneous media as follows

$$\frac{\mathrm{d}U_{mn}^{\mathrm{P}(K)\parallel}}{\mathrm{d}\tau} + \frac{U_{mn}^{\mathrm{P}(K)\parallel}}{2J}\frac{\mathrm{d}J}{\mathrm{d}\tau} = \frac{1}{2}\{L_{in}(U_{kn}^{\mathrm{P}(K-1)}) - M_{in}(U_{kn}^{\mathrm{P}(K)\perp})\}g_i^{\mathrm{P}} g_m^{\mathrm{P}} . \tag{7}$$

Taking into account that the Jacobian J in Eq. (7) can be expressed for point sources in homogeneous media in a simple form (see Appendix A)

$$J = \nu^3 \tau^2 \sin\vartheta ,$$

where ϑ is the angle between a ray and the vertical and ν is the group velocity, we arrive at the transport equation which reads

$$\frac{\mathrm{d}U_{mn}^{\mathrm{P}(K)\parallel}}{\mathrm{d}\tau} + \frac{U_{mn}^{\mathrm{P}(K)\parallel}}{\tau} = \frac{1}{2}\{L_{in}(U_{kn}^{\mathrm{P}(K-1)}) - M_{in}(U_{kn}^{\mathrm{P}(K)\perp})\}g_i^{\mathrm{P}} g_m^{\mathrm{P}} . \tag{8}$$

Eq. (8) is the ordinary differential inhomogeneous equation of the first order. The zeroth-order term is a solution of the respective homogeneous equation, the right-hand side of Eq. (8) being zero for $K = \vartheta$. The solution has a form

$$U_{mn}^{\mathrm{P}(0)\parallel} = \frac{R_{mn}^{\mathrm{P}(0)\parallel}(\vartheta, \varphi)}{\tau}, \tag{9}$$

where $R_{mn}^{\mathrm{P}(0)\parallel}(\vartheta, \varphi)$ is an integration constant being an arbitrary function of ϑ and φ. Physically $R_{mn}^{\mathrm{P}(0)\parallel}(\vartheta, \varphi)$ represents the far-field P-wave radiation pattern of a point single force source in the zeroth-order ray approximation.

The principal component in the first-order approximation is a solution of Eq. (8) for $K = 1$. Inserting Eqs. (9) and (5) into Eq. (6) it can be shown that $U_{kn}^{\mathrm{P}(1)\perp}$ depends on τ as $1/\tau^2$, and consequently, that the right-hand side in Eq. (8) for $K = 1$ depends on τ as $1/\tau^3$. Using for the right-hand of Eq. (8) the following tentative notation

$$\frac{f_{mn}^{\mathrm{P}(1)}(\vartheta, \varphi)}{\tau^3} \equiv \frac{1}{2}\{L_{in}(U_{kn}^{\mathrm{P}(0)}) - M_{in}(U_{kn}^{\mathrm{P}(1)\perp})\}g_i^{\mathrm{P}} g_m^{\mathrm{P}} , \tag{10}$$

the first-order transport equation and its solution can be written as follows

$$\frac{\mathrm{d}U_{mn}^{\mathrm{P}(1)\parallel}}{\mathrm{d}\tau} + \frac{U_{mn}^{\mathrm{P}(1)\parallel}}{\tau} = \frac{f_{mn}^{\mathrm{P}(1)}(\vartheta, \varphi)}{\tau^3}, \quad U_{mn}^{\mathrm{P}(1)\parallel} = \frac{R_{mn}^{\mathrm{P}(1)\parallel}(\vartheta, \varphi)}{\tau^2} + \frac{C_{mn}^{\mathrm{P}(1)\parallel}(\vartheta, \phi)}{\tau}, \tag{11}$$

where

$$R_{mn}^{\mathrm{P}(1)\parallel}(\vartheta,\varphi) = -f_{mn}^{\mathrm{P}(1)}(\vartheta,\varphi) = -\frac{\tau^3}{2}\{L_{in}(U_{kn}^{\mathrm{P}(0)}) - M_{in}(U_{kn}^{\mathrm{P}(1)\perp})\}g_i^{\mathrm{P}}g_m^{\mathrm{P}} \tag{12}$$

is the radiation pattern of the P-wave principal component in the first-order ray approximation. The function $C_{mn}^{\mathrm{P}(1)\parallel}(\vartheta,\varphi)$ is an arbitrary integration constant, since a general solution should involve also any solution of the homogeneous equation. The function $C_{mn}^{\mathrm{P}(1)\parallel}(\vartheta,\varphi)$ affects the far-field wave radiation of the source in such a way that not only the amplitude but also the shape of the radiated far-field wave is directionally dependent. Since we shall focus only on *sources generating waves with the waveforms in the far field directionally independent* (see formulae (17) and (18)), the function $C_{mn}^{\mathrm{P}(1)\parallel}(\vartheta,\varphi)$ is identically zero in this case. Analogously, we can write for the Kth-order ray approximation $(K>1)$

$$\frac{\mathrm{d}U_{mn}^{\mathrm{P}(K)\parallel}}{\mathrm{d}\tau} + \frac{U_{mn}^{\mathrm{P}(K)\parallel}}{\tau} = \frac{f_{mn}^{\mathrm{P}(K)}(\vartheta,\varphi)}{\tau^{K+2}}, \tag{13}$$

where we used again a tentative notation for the right-hand side of Eq. (8) expressing the dependence on τ explicitly. The solution of Eq. (13) is as follows

$$U_{mn}^{\mathrm{P}(K)\parallel} = \frac{R_{mn}^{\mathrm{P}(K)\parallel}(\vartheta,\varphi)}{\tau^{K+1}} + \frac{C_{mn}^{\mathrm{P}(K)\parallel}(\vartheta,\varphi)}{\tau}, \tag{14}$$

where

$$R_{mn}^{\mathrm{P}(K)\parallel}(\vartheta,\varphi) = -\frac{1}{K}f_{mn}^{\mathrm{P}(K)}(\vartheta,\varphi) = -\frac{\tau^{K+2}}{2K}\{L_{in}(U_{kn}^{\mathrm{P}(K-1)}) - M_{in}(U_{kn}^{\mathrm{P}(K)\perp})\}g_i^{\mathrm{P}}g_m^{\mathrm{P}} \tag{15}$$

is the radiation pattern of the P-wave principal component in the Kth-order ray approximation and $C_{mn}^{\mathrm{P}(K)\parallel}(\vartheta,\varphi)$ is an arbitrary integration constant. Similarly to the first-order ray approximation, we conclude that $C_{mn}^{\mathrm{P}(K)\parallel}(\vartheta,\varphi)$ is identically zero. From Eqs. (14) and (15), we arrive at an explicit general formula for the principal component of the Kth-order ray approximation for P waves $(K>0)$

$$U_{mn}^{\mathrm{P}(K)\parallel} = \frac{\tau}{2K}\{M_{in}(U_{kn}^{\mathrm{P}(K)\perp}) - L_{in}(U_{kn}^{\mathrm{P}(K-1)})\}g_i^{\mathrm{P}}g_m^{\mathrm{P}}. \tag{16}$$

From formulae (5), (6) and (16) we can see that all the higher-order ray approximations can be obtained only by the differentiation of the zeroth-order term, that is a mathematically elementary procedure. It should be noted, however, that this procedure is numerically unstable and analytically tends to produce rather extensive formulae.

3. Green tensor for anisotropic media

An exact solution of the Green tensor for homogeneous anisotropic media was found by Buchwald [5], Lighthill [16] and Burridge [6]. It has three parts corresponding to P, S1 and S2 waves. Under the far-field approximation, each part $G_{in}^{\mathrm{P}}(x_j,t)$, $G_{in}^{\mathrm{S1}}(x_j,t)$ and $G_{in}^{\mathrm{S2}}(x_j,t)$ of the Green tensor can be explicitly expressed in terms of the Gaussian curvature of the slowness surface $\bar{K}$ and the group velocity ν [6,15]

$$G_{in}(\boldsymbol{x},t) = \frac{1}{4\pi\rho}\frac{1}{\bar{K}^{1/2}\nu^2}\frac{g_i g_n}{\tau}\delta(t-\tau(\boldsymbol{x})), \tag{17}$$

where we omitted superscripts denoting the particular wavetype. Alternatively in terms of the Gaussian curvature of the wave surface (group velocity surface) K^* and slowness p

$$G_{in}(x_j,t) = \frac{1}{4\pi\rho}K^{*1/2}p^2\frac{g_i g_n}{\tau}\delta(t-\tau(x_j)). \tag{18}$$

Formula (18) can be obtained from formula (12) of Ben-Menahem et al. [4] by applying an obvious relation between Gaussian curvature of the wavefront K^{**} and Gaussian curvature of the wave surface K^* ($K^{**}=K^*\tau^2$) or from formula (17) by taking the following identity into account (see Appendix B for a derivation):

$$K^*\bar{K}=\frac{1}{p^4\nu^4}. \tag{19}$$

Formulae (17) and (18) fail in the vicinity of parabolic points at which the Gaussian curvature of the slowness surface is zero. These cases require a more careful analysis (see Burridge [6]) related to triplication of the wave surface and will not be considered in this paper.

4. SH-wave Green tensor for a transversely isotropic medium

4.1. The zeroth-order term

Next, we will consider a transversely isotropic medium (hereafter called TI). Without loss of generality we will assume the vertical axis of symmetry. The Christoffel tensor reads

$$\begin{gathered}\Gamma_{11}=a_{11}p_1^2+a_{66}p_2^2+a_{44}p_3^2,\quad \Gamma_{22}=a_{66}p_1^2+a_{11}p_2^2+a_{44}p_3^2,\quad \Gamma_{33}=a_{44}(p_1^2+p_2^2)+a_{33}p_3^2,\\ \Gamma_{23}=(a_{13}+a_{44})p_2p_3,\quad \Gamma_{12}=(a_{11}-a_{66})p_1p_2,\quad \Gamma_{13}=(a_{13}+a_{44})p_1p_3,\end{gathered} \tag{20}$$

where a_{ij} is the conventionally used 2-index notation of tensor a_{ijkl} (see Musgrave [19], formula (3.13.4)). Consequently, eigenvalues and eigenvectors of Γ_{ij} are expressed as follows

$$\begin{gathered}G^{\mathrm{P,SV}}=\frac{1}{2}\left[a_{11}(p_1^2+p_2^2)+a_{33}p_3^2+a_{44}p^2\pm\sqrt{Ap^4+Bp^2p_3^2+Cp_3^4}\right],\\ G^{\mathrm{SH}}=a_{66}(p_1^2+p_2^2)+a_{44}p_3^2,\\ \boldsymbol{g}^{\mathrm{P}}=\begin{bmatrix}\sin\theta\cos\varphi\\ \sin\theta\sin\varphi\\ \cos\theta\end{bmatrix},\quad \boldsymbol{g}^{\mathrm{SV}}=\begin{bmatrix}\cos\theta\cos\varphi\\ \cos\theta\sin\varphi\\ -\sin\theta\end{bmatrix},\quad \boldsymbol{g}^{\mathrm{SH}}=\begin{bmatrix}\sin\varphi\\ -\cos\varphi\\ 0\end{bmatrix},\end{gathered} \tag{21}$$

where

$$\begin{gathered}p^2=p_1^2+p_2^2+p_3^2,\quad A=(a_{11}-a_{44})^2,\\ B=(-2a_{11}^2-2a_{11}a_{33}+6a_{11}a_{44}+2a_{33}a_{44}+4a_{13}^2+8a_{13}a_{44}),\\ C=(a_{11}^2+2a_{11}a_{33}+a_{33}^2-4a_{11}a_{44}-4a_{33}a_{44}-4a_{13}^2-8a_{13}a_{44}),\\ \sin\theta=\sqrt{\frac{G^{\mathrm{P}}-\Gamma_{33}}{G^{\mathrm{P}}-G^{\mathrm{SV}}}},\quad \cos\theta=\sqrt{\frac{G^{\mathrm{P}}-\bar{\Gamma}_{11}}{G^{\mathrm{P}}-G^{\mathrm{SV}}}},\quad \bar{\Gamma}_{11}=a_{11}(p_1^2+p_2^2)+a_{44}p_3^2.\end{gathered} \tag{22}$$

Emphasize that orthogonality of polarization vectors $\boldsymbol{g}^{\mathrm{P}}$ and $\boldsymbol{g}^{\mathrm{SV}}$ in Eq. (21) holds for a selected slowness vector, but not for a ray. According to Kendall et al. [15], the Gaussian curvature of the slowness surface is calculated by

$$\begin{aligned}\bar{K}=\frac{1}{\nu^4}\{&\nu_1^2(\nu_{2,2}\nu_{3,3}-\nu_{2,3}^2)+2\nu_2\nu_3(\nu_{1,2}\nu_{1,3}-\nu_{1,1}\nu_{2,3})+\nu_2^2(\nu_{3,3}\nu_{1,1}-\nu_{3,1}^2)+2\nu_3\nu_1(\nu_{2,3}\nu_{2,1}-\nu_{2,2}\nu_{3,1})\\ &+\nu_3^2(\nu_{1,1}\nu_{2,2}-\nu_{1,2}^2)+2\nu_1\nu_2(\nu_{3,1}\nu_{3,2}-\nu_{3,3}\nu_{1,2})\},\end{aligned} \tag{23}$$

where $\nu_{i,j} = \partial\nu_i / \partial p_j$. Specifying for SH-wave in TI

$$\nu^{SH} = \frac{1}{2}\frac{\partial G^{SH}}{\partial \boldsymbol{p}} = \begin{bmatrix} a_{66}p_1 \\ a_{66}p_2 \\ a_{44}p_3 \end{bmatrix} \quad \text{and} \quad \bar{K}^{SH} = \frac{1}{\nu^4} a_{66}^2 a_{44} . \tag{24}$$

Inserting formula (24) into Eq. (17) we can express the zeroth-order approximation of the SH-wave Green tensor

$$G_{in}^{SH(0)}(x_j, t) = \frac{1}{4\pi\rho}\frac{1}{a_{66}\sqrt{a_{44}}}\frac{g_i^{SH} g_n^{SH}}{\tau}\,\delta(t-\tau(x_j)) . \tag{25}$$

4.2. The first-order term

For the *additional component* of the first-order term formula (6) yields (see Appendix C for details)

$$U_{mn}^{SH(1)\perp} = M_{in}(U_{kn}^{SH(0)})\left\{\frac{g_i^P g_m^P}{G^P - G^{SH}} + \frac{g_i^{SV} g_m^{SV}}{G^{SV} - G^{SH}}\right\} = -\frac{1}{4\pi\rho\sqrt{a_{44}}}\frac{g_m^{SH\perp} g_n^{SH\perp}}{r^2 \sin^2\vartheta} , \tag{26}$$

where $\boldsymbol{g}^{SH\perp} = \begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{bmatrix}$,

ϑ denotes the angle between a ray and the vertical, and r is the distance of an observation point from the source. The eigenvalue G^{SH} equals 1.

According to Eq. (16), we can write for the principal component $U_{mn}^{SH(1)\parallel}$

$$U_{mn}^{SH(1)\parallel} = \frac{\tau}{2}\{M_{in}(U_{kn}^{SH(1)\perp}) - L_{in}(U_{kn}^{SH(0)})\} g_i^{SH} g_m^{SH} = \frac{1}{4\pi\rho\sqrt{a_{44}}}\frac{g_m^{SH} g_n^{SH}}{r^2 \sin^2\vartheta} . \tag{27}$$

The complete first-order ray amplitude $U_{mn}^{SH(1)}$ is finally written as

$$U_{mn}^{SH(1)} = \frac{1}{4\pi\rho\sqrt{a_{44}}}\frac{g_m^{SH} g_n^{SH} - g_m^{SH\perp} g_n^{SH\perp}}{r^2 \sin^2\vartheta} . \tag{28}$$

4.3. Complete formula for the Green tensor

Calculating higher-order terms, we substitute formulae (25) and (28) into Eqs. (6) and (16). We then reach a surprising result that the second-order term and all the higher terms equal zero. Thus the complete SH-wave Green tensor is expressed in a very simple following form

$$G_{in}^{SH}(x_j, t) = \frac{1}{4\pi\rho}\frac{1}{\sqrt{a_{44}}}\left\{\frac{\delta(t-\tau)}{a_{66}\tau} g_i^{SH} g_n^{SH} + \frac{H(t-\tau)}{r^2 \sin^2\vartheta}(g_i^{SH} g_n^{SH} - g_i^{SH\perp} g_n^{SH\perp})\right\} , \tag{29}$$

where $\tau = r\sqrt{\frac{\sin^2\vartheta}{a_{66}} + \frac{\cos^2\vartheta}{a_{44}}}$.

$H(t)$ denotes the Heaviside step function, $r = |\boldsymbol{x}|$ is the distance of an observation point from the source, ϑ is the angle between a ray and the vertical, τ is the traveltime, and $\boldsymbol{g}^{SH\perp}$ is defined in formula (26). Formula (29) coincides with an exact formula obtained by Ben-Menahem and Sena [3] derived in frequency domain. The first term of formula (29) describes the far-field wave, the second term describes the near-field wave.

5. Discussion

We introduce tensor equations of the ray theory for homogeneous anisotropic media. For these media, we solved the higher-order transport equations explicitly and we obtained higher-order ray approximations only by the differentiation of lower-order terms. For exemplifying analytical calculations of higher-order ray approximations, we choose the SH-wave Green tensor for transversely isotropic media. For TI, the SH-wave is simple forming an ellipsoidal wavefront with no caustics and displaying a polarization perpendicular to a symmetry axis. We show that the final ray theoretical formula (29) coincides with an exact formula derived by Ben-Menahem and Sena [3]. The complete ray theoretical Green function involves only the zeroth- and first-order terms. All the higher-order ray approximations are zero. Surprisingly, the SH-wave Green tensor for TI is even simpler than the S-wave Green tensor in isotropy, which involves also the non-zero second-order term (see Vavryčuk and Yomogida [28]). In order to understand this seeming incompatibility, we should decompose the S waves in isotropic media into SH and SV waves (polarization of SH wave being in the horizontal plane) and to calculate the higher-order ray approximations for SV and SH waves separately. It can be shown that the ray expansion of the SV wave in isotropy consists of three non-zero terms, but the ray expansion of the SH wave has likewise in TI two non-zero terms only.

The simplicity of the Green tensor will probably be lost for P- and SV-waves in TI media or for waves in more general media than TI. The reason is a more complex directional variation of polarization vectors and a more complex shape of wavefronts where also complications due to caustics can arise. It is likely that the complete ray theoretical Green function will involve infinite number of non-zero higher approximations and it will not necessarily converge to an exact formula. Nevertheless, we assume that a further development of the proposed approach is promising and its application to more general cases than that presented here will be useful. Apparently, such calculations, which are mathematically elementary, will need, however, a sophisticated elaboration with rather extensive analytical formulae.

Appendix A. Jacobian determination

In this appendix, we derive the Jacobian of the transformation from the Cartesian to the ray coordinates for a point source in a homogeneous anisotropic medium. Since rays are straight lines in the homogeneous medium, we use traveltime τ and take-off angles ϑ, φ of a ray as the ray coordinates. Emphasize that we use the take-off angles of a ray, but not the take-off angles of a slowness vector, that are in standard use. The forward and inverse transformations from the Cartesian coordinates x_1, x_2, x_3 to the ray coordinates τ, ϑ, φ take the form

$$x_1 = \tau\nu(\vartheta,\varphi)\sin\vartheta\cos\varphi\,, \quad x_2 = \tau\nu(\vartheta,\varphi)\sin\vartheta\sin\varphi\,, \quad x_3 = \tau\nu(\vartheta,\varphi)\cos\vartheta\,,$$

$$\tau = \frac{\sqrt{x_1^2+x_2^2+x_3^2}}{\nu}\,, \quad \varphi = \arctan\frac{x_2}{x_1}\,, \quad \vartheta = \arctan\frac{\sqrt{x_1^2+x_2^2}}{x_3}\,,$$

where ν is the group velocity. Such transformations exist even in the case of triplicated wavefronts, but applied only locally for a particular branch and with except of caustic points. The Jacobian J is determined by

$$J \equiv \begin{vmatrix} \dfrac{\partial x_1}{\partial \tau} & \dfrac{\partial x_1}{\partial \vartheta} & \dfrac{\partial x_1}{\partial \varphi} \\ \dfrac{\partial x_2}{\partial \tau} & \dfrac{\partial x_2}{\delta \vartheta} & \dfrac{\partial x_2}{\partial \varphi} \\ \dfrac{\partial x_3}{\partial \tau} & \dfrac{\partial x_3}{\partial \vartheta} & \dfrac{\partial x_3}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \nu\sin\vartheta\cos\varphi & \tau\cos\varphi\left(\nu\cos\vartheta + \dfrac{\partial\nu}{\partial\vartheta}\sin\vartheta\right) & \tau\sin\vartheta\left(-\nu\sin\varphi + \dfrac{\partial\nu}{\partial\varphi}\cos\varphi\right) \\ \nu\sin\vartheta\sin\varphi & \tau\sin\varphi\left(\nu\cos\vartheta + \dfrac{\partial\nu}{\partial\vartheta}\sin\vartheta\right) & \tau\sin\vartheta\left(\nu\cos\varphi + \dfrac{\partial\nu}{\partial\varphi}\sin\varphi\right) \\ \nu\cos\vartheta & \tau\left(-\nu\sin\vartheta + \dfrac{\partial\nu}{\partial\vartheta}\cos\vartheta\right) & \tau\dfrac{\partial\nu}{\partial\varphi}\cos\vartheta \end{vmatrix}$$

$$= \nu^3\tau^2\sin\vartheta\,. \tag{A.1}$$

Analogously, the inverse transformation gives

$$J^{-1} \equiv \begin{vmatrix} \frac{\partial \tau}{\partial x_1} & \frac{\partial \tau}{\partial x_2} & \frac{\partial \tau}{\partial x_3} \\ \frac{\partial \vartheta}{\partial x_1} & \frac{\partial \vartheta}{\partial x_2} & \frac{\partial \vartheta}{\partial x_3} \\ \frac{\partial \varphi}{\partial x_1} & \frac{\partial \varphi}{\partial x_2} & \frac{\partial \varphi}{\partial x_3} \end{vmatrix} = \begin{vmatrix} p_1 & p_2 & p_3 \\ \frac{\cos\vartheta\cos\varphi}{r} & \frac{\cos\vartheta\sin\varphi}{r} & -\frac{\sin\vartheta}{r} \\ -\frac{\sin\varphi}{r\sin\vartheta} & \frac{\cos\varphi}{r\sin\vartheta} & 0 \end{vmatrix} = \frac{1}{\nu r^2 \sin\vartheta}, \tag{A.2}$$

where $\boldsymbol{p}$ is the slowness vector, $r=\sqrt{x_1^2+x_2^2+x_3^2}=\tau\nu$ is the distance from the source. Here we used the following identity

$$\frac{1}{\nu} = p_1 \sin\vartheta\cos\varphi + p_2 \sin\vartheta\sin\varphi + p_3\cos\vartheta .$$

Appendix B. Gaussian curvatures of the slowness and wave surfaces

Let us define a surface by equation $\boldsymbol{x}=\boldsymbol{x}(\gamma_1, \gamma_2)$, where γ_1, γ_2 are the curvilinear coordinates of a point of the surface. Gaussian curvature K is defined as follows (Struik [23], p. 156)

$$K = \frac{\left(\frac{\partial \boldsymbol{n}}{\partial \gamma_1} \times \frac{\partial \boldsymbol{n}}{\partial \gamma_2}\right)\cdot\left(\frac{\partial \boldsymbol{x}}{\partial \gamma_1} \times \frac{\partial \boldsymbol{x}}{\partial \gamma_2}\right)}{\left|\frac{\partial \boldsymbol{x}}{\partial \gamma_1} \times \frac{\partial \boldsymbol{x}}{\partial \gamma_2}\right|^2}, \tag{B.1}$$

where $\boldsymbol{n}$ denotes the unit normal to the surface. Since the slowness vector $\boldsymbol{p}$ is the normal to the wave surface $\boldsymbol{\nu}=\boldsymbol{\nu}(\gamma_1, \gamma_2)$ and the group velocity vector $\boldsymbol{\nu}$ is the normal to the slowness surface $\boldsymbol{p}=\boldsymbol{p}(\gamma_1, \gamma_2)$ (Musgrave [19], p. 79), we can write

$$\bar{K} = \frac{1}{\nu^2}\frac{\left(\frac{\partial \boldsymbol{\nu}}{\partial \gamma_1} \times \frac{\partial \boldsymbol{\nu}}{\partial \gamma_2}\right)\cdot\left(\frac{\partial \boldsymbol{p}}{\partial \gamma_1} \times \frac{\partial \boldsymbol{p}}{\partial \gamma_2}\right)}{\left|\frac{\partial \boldsymbol{p}}{\partial \gamma_1} \times \frac{\partial \boldsymbol{p}}{\partial \gamma_2}\right|^2}, \quad K^* = \frac{1}{p^2}\frac{\left(\frac{\partial \boldsymbol{p}}{\partial \gamma_1} \times \frac{\partial \boldsymbol{p}}{\partial \gamma_2}\right)\cdot\left(\frac{\partial \boldsymbol{\nu}}{\partial \gamma_1} \times \frac{\partial \boldsymbol{\nu}}{\partial \gamma_2}\right)}{\left|\frac{\partial \boldsymbol{\nu}}{\partial \gamma_1} \times \frac{\partial \boldsymbol{\nu}}{\partial \gamma_2}\right|^2}, \tag{B.2}$$

where $\bar{K}$ and K^* denote the Gaussian curvatures of the slowness and wave surfaces, respectively. Taking into account that

$$\left(\frac{\partial \boldsymbol{\nu}}{\partial \gamma_1} \times \frac{\partial \boldsymbol{\nu}}{\partial \gamma_2}\right)\cdot\left(\frac{\partial \boldsymbol{p}}{\partial \gamma_1} \times \frac{\partial \boldsymbol{p}}{\partial \gamma_2}\right) = \left|\frac{\partial \boldsymbol{\nu}}{\partial \gamma_1} \times \frac{\partial \boldsymbol{\nu}}{\partial \gamma_2}\right| \left|\frac{\partial \boldsymbol{p}}{\partial \gamma_1} \times \frac{\partial \boldsymbol{p}}{\partial \gamma_2}\right| \cos\delta, \tag{B.3}$$

where δ is the angle between the group velocity vector and the slowness vector, we arrive at the equation

$$K^*\bar{K} = \frac{\cos^2\delta}{\nu^2 p^2}. \tag{B.4}$$

Applying the following formula

$$\boldsymbol{\nu}\cdot\boldsymbol{p} = \nu p \cos\delta = 1, \tag{B.5}$$

we get

$$K^*\bar{K} = \cos^4\delta = \frac{1}{p^4\nu^4}. \tag{B.6}$$

Appendix C. Auxiliary formulae

In this appendix, some details about the calculation of the first-order ray approximation of the SH-wave Green tensor in a homogeneous transversely isotropic medium are given. Evaluating differential operators $M_{in}(U_{kn}^{\mathrm{SH}(K)})$ and $L_{in}(U_{kn}^{\mathrm{SH}(K)})$, we need the following relations and derivatives

$$\frac{\partial\tau}{\partial \boldsymbol{x}} = \boldsymbol{p} = \begin{bmatrix} p\sin\bar{\vartheta}\cos\varphi \\ p\sin\bar{\vartheta}\sin\varphi \\ p\cos\bar{\vartheta} \end{bmatrix}, \quad p = |\boldsymbol{p}| = \frac{1}{\sqrt{a_{66}\sin^2\bar{\vartheta} + a_{44}\cos^2\bar{\vartheta}}},$$

$$\begin{bmatrix} \dfrac{\partial\varphi}{\partial x_1} & \dfrac{\partial\bar{\vartheta}}{\partial x_1} & \dfrac{\partial p}{\partial x_1} \\ \dfrac{\partial\varphi}{\partial x_2} & \dfrac{\partial\bar{\vartheta}}{\partial x_2} & \dfrac{\partial p}{\partial x_2} \\ \dfrac{\partial\varphi}{\partial x_3} & \dfrac{\partial\bar{\vartheta}}{\partial x_3} & \dfrac{\partial p}{\partial x_3} \end{bmatrix} = \begin{bmatrix} -\dfrac{\sin\varphi}{a_{66}\tau p\sin\bar{\vartheta}} & \dfrac{\cos\bar{\vartheta}\cos\varphi}{a_{66}\tau p} & \dfrac{a_{44}-a_{66}}{a_{66}}\dfrac{\sin\bar{\vartheta}\cos^2\bar{\vartheta}\cos\varphi}{\tau}p^2 \\ \dfrac{\cos\varphi}{a_{66}\tau p\sin\bar{\vartheta}} & \dfrac{\cos\bar{\vartheta}\sin\varphi}{a_{66}\tau p} & \dfrac{a_{44}-a_{66}}{a_{66}}\dfrac{\sin\bar{\vartheta}\cos^2\bar{\vartheta}\sin\varphi}{\tau}p^2 \\ 0 & -\dfrac{\sin\bar{\vartheta}}{a_{44}\tau p} & -\dfrac{a_{44}-a_{66}}{a_{66}}\dfrac{\sin^2\bar{\vartheta}\cos\bar{\vartheta}}{\tau}p^2 \end{bmatrix}, \tag{C.1}$$

where $\bar{\vartheta}$ denotes the angle between slowness vector $\boldsymbol{p}$ and the vertical axis.

Evaluating $M_{in}(U_{kn}^{\mathrm{SH}(0)})$, we get

$$M_{in}(U_{kn}^{\mathrm{SH}(0)}) = \frac{C}{a_{66}\tau^2\sin\bar{\vartheta}}\begin{bmatrix} A\sin\bar{\vartheta}\cos^2\varphi & A\sin\bar{\vartheta}\sin\varphi\cos\varphi & 0 \\ A\sin\bar{\vartheta}\sin\varphi\cos\varphi & A\sin\bar{\vartheta}\sin^2\varphi & 0 \\ -B\cos\bar{\vartheta}\cos\varphi & -B\cos\bar{\vartheta}\sin\varphi & 0 \end{bmatrix}, \tag{C.2}$$

where $A = a_{66} - a_{11}$, $B = a_{13} + a_{44}$ and $C = (1/4\pi\rho)(1/a_{66}\sqrt{a_{44}})$.

Inserting formula (C.2) into Eq. (6), we get the additional component

$$U_{mn}^{\mathrm{SH}(1)\perp} = -\frac{C}{a_{66}\tau^2p^2\sin^2\bar{\vartheta}}\begin{bmatrix} \cos^2\varphi & \sin\varphi\cos\varphi & 0 \\ \sin\varphi\cos\varphi & \sin^2\varphi & 0 \\ 0 & 0 & 0 \end{bmatrix} = -\frac{a_{66}C}{r^2\sin^2\vartheta}g_m^{\mathrm{SH}\perp}g_n^{\mathrm{SH}\perp}, \tag{C.3}$$

where we used

$$\sin^2\theta = \frac{G^{\mathrm{P}} - \bar{\Gamma}_{33}}{G^{\mathrm{P}} - G^{\mathrm{SV}}}, \quad \cos^2\theta = \frac{G^{\mathrm{P}} - \bar{\Gamma}_{11}}{G^{\mathrm{P}} - G^{\mathrm{SV}}}, \quad \sin\theta\cos\theta = \frac{\bar{\Gamma}_{13}}{G^{\mathrm{P}} - G^{\mathrm{SV}}},$$

$$a_{66}^2\tau^2p^2\sin^2\bar{\vartheta} = r^2\sin^2\vartheta \quad \text{and} \quad \boldsymbol{g}^{\mathrm{SH}\perp} = \begin{bmatrix} \cos\varphi \\ \sin\varphi \\ 0 \end{bmatrix}.$$

$\bar{\Gamma}_{ik}$ denotes the Christoffel tensor (Eq. (20)) in the x_1–x_3 plane, ϑ denotes the angle between a ray and the vertical axis, θ is the angle between polarization vector $\boldsymbol{g}^{\mathrm{P}}$ and the vertical axis, and r is the distance from the source.

Operators $M_{in}(U_{kn}^{\mathrm{SH}(1)\perp})$ and $L_{in}(U_{kn}^{\mathrm{SH}(0)})$ are evaluated in a similar way as $M_{in}(U_{kn}^{\mathrm{SH}(0)})$ having the final form as follows:

$$M_{in}(U_{kn}^{\mathrm{SH}(1)\perp})g_i^{\mathrm{SH}} = \frac{C(a_{66}-a_{11})}{a_{66}^2\tau^3p^2\sin^2\bar{\vartheta}}\,g_n^{\mathrm{SH}}, \quad L_{in}(U_{kn}^{\mathrm{SH}(0)})g_i^{\mathrm{SH}} = -\frac{C(a_{11}+a_{66})}{a_{66}^2\tau^3p^2\sin^2\bar{\vartheta}}\,g_n^{\mathrm{SH}}, \tag{C.4}$$

where C is defined in formula (C.2).

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