# Inversion for weak triclinic anisotropy from acoustic axes 

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#### Abstract

Acoustic axes are directions in anisotropic elastic media, in which phase velocities of two or three plane waves ( $P, S 1$ or $S 2$ waves) coincide. Acoustic axes are important, because they can cause singularities in the field of polarization vectors and anomalies in the shape of the slowness surface. The maximum number of acoustic axes in triclinic anisotropy is 16 , and their directions depend on anisotropy parameters in a complicate way. Under weak anisotropy approximation this dependence simplifies and the directions of acoustic axes can be used for the inversion for anisotropy parameters. The maximum acoustic axes under weak anisotropy is 16 , the minimum number of acoustic axes is zero. In the inversion, we can retrieve 13 combinations of anisotropy parameters provided we use directions of 7 acoustic axes at least. Under weak anisotropy approximation, the directions of acoustic axes are insensitive to strength of anisotropy; hence we cannot invert for absolute values of weak anisotropy parameters, but only for their relative values. Numerical tests have shown that the inversion is applicable only to very weak anisotropy with strength of less than $5 \%$, provided that the acoustic axes used in the inversion are determined with an accuracy of $0.1^{\circ}$ or better. In this case the inversion yields an average error for elastic parameters of less than $10 \%$. In order to invert for the total set of 21 anisotropy parameters it is necessary to combine the measurements of the directions of the acoustic axes with measurements of other attributes of elastic waves in anisotropic media.


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## 1. Introduction

Acoustic axes (singularities, degeneracies) are directions in anisotropic media, in which phase velocities of two or three plane waves ( $P, S 1$ or $S 2$ waves) coincide. Such directions are very important, because they can cause singularities in the field of polarization vectors (see Fig. 1) and anomalies in the shape of the slowness surface [1-5]. Acoustic axes are frequently associated with caustics and anti-caustics on the wave surface [6-8], with triplications and with energy focusing [7,9-11]. They also pose complications in tracing rays [12,13] and in wavefield modelling because of the coupling of waves [14-19].

We distinguish several types of acoustic axes [20-24]: they can form either single isolated points classified as kiss, conical or wedge singularities, or they can combine into line singularities. The most typical acoustic axis in triclinic anisotropy is the conical singularity. This singularity is stable, while the other singularities are unstable. The stable singularity is defined as the singularity, which can change slightly its direction but it cannot split or disappear under any small perturbation of elastic parameters. The unstable singularity is split into stable singularities or disappears under any small perturbation of elastic parameters. The maximum number of isolated acoustic axes in triclinic anisotropy is 16 [25-27] as in monoclinic and orthorhombic symmetries [28]. The directions are calculated by solving two coupled polynomial equations of the sixth order in two variables [20,23,27]. Obviously this implies that the positions of acoustic axes are complicated functions of anisotropy parameters.

[^0]

Fig. 1. The field of polarization vectors on the slowness surfaces near a conical singularity for the $S 1$ and $S 2$ waves. The polarization vectors are projected into the plane perpendicular to the polarization vector of the non-degenerate wave in the singularity. The topological charge of the singularity is $-1 / 2$. The dot marks the position of the singularity.

Properties of elastic waves in anisotropic media often simplify under the assumption of weak anisotropy [8,29-34]. Under weak anisotropy, the shapes of slowness and wave surfaces and the behaviour of waves are usually much simpler and modelling of propagating waves is easier than in strong anisotropy. The weak anisotropy is also a reasonable assumption valid for many real materials including most rocks and geological structures in the Earth [29,35,36]. As regards the acoustic axes, Vavryčuk [37] has shown that surprisingly their calculation remains complicated also under weak triclinic anisotropy. The maximum number of acoustic axes remains 16 as in strong triclinic anisotropy. The only difference compared with strong triclinic anisotropy is that instead of solving two coupled equations of the sixth order it is sufficient to solve two coupled equations of the fifth order. This somewhat simplifies the problem and speeds up the calculations, though not substantially. Although the weak anisotropy approximation does not simplify the problem of calculating the acoustic axes significantly, it can find some useful applications. In this paper, I shall show that the weak anisotropy approximation can be particularly advantageous in the inversion for anisotropy parameters from the directions of the acoustic axes.

## 2. Calculation of acoustic axes in strong triclinic anisotropy

The Christoffel tensor $\boldsymbol{\Gamma}(\mathbf{n})$ is defined as $[22,38,39]$

$$
\begin{equation*}
\Gamma_{j k}(\mathbf{n})=a_{i j k l} n_{i} n_{l}, \tag{1}
\end{equation*}
$$

where $a_{i j k l}$ are the density-normalized elastic parameters and $\mathbf{n}$ is the unit vector defining the slowness direction. The Einstein summation convention over repeated subscripts is applied. For physically realizable media, the elastic parameters $a_{i j k l}$ satisfy the stability conditions (Helbig [24, Eqs. (5.5)-(5.10)]) and the Christoffel tensor $\boldsymbol{\Gamma}(\mathbf{n})$ is positive-definite for all directions $\mathbf{n}$. The Christoffel tensor $\boldsymbol{\Gamma}(\mathbf{n})$ has three eigenvalues $G^{(M)}$ and three unit eigenvectors $\mathbf{g}^{(M)}$, which are calculated from

$$
\begin{equation*}
\Gamma_{j k} g_{k}^{(M)}=G^{(M)} g_{j}^{(M)}, \quad M=1,2,3, \tag{2}
\end{equation*}
$$

where $M$ denotes the type of wave ( $P, S 1$ or $S 2$ ). The eigenvalue corresponds to the squared phase velocity, $G=c^{2}$, and the eigenvector defines the polarization vector of the wave.

Acoustic axes are directions, in which two eigenvalues of $\Gamma(\mathbf{n})$ coincide

$$
\begin{equation*}
G^{(1)}(\mathbf{n}) \neq G^{(2)}(\mathbf{n})=G^{(3)}(\mathbf{n}) \tag{3}
\end{equation*}
$$

Exceptionally, all three eigenvalues can coincide, but such acoustic axis is very rare and it will not be considered here. Using the spectral decomposition of $\boldsymbol{\Gamma}(\mathbf{n})$ and applying the condition for the acoustic axis (3), we obtain [40]

$$
\begin{equation*}
\Gamma_{j k}=\left(G^{(1)}-G^{(2)}\right) g_{j}^{(1)} g_{k}^{(1)}+G^{(2)} \delta_{j k}, \tag{4}
\end{equation*}
$$

and subsequently

$$
\begin{equation*}
a_{i j k l} s_{i} s_{l}=g_{j} g_{k}+\delta_{j k}, \tag{5}
\end{equation*}
$$

where $\mathbf{s}=\mathbf{n} / \sqrt{G^{(2)}}$ is the slowness vector of the degenerate wave and $\mathbf{g}=\mathbf{g}^{(1)} \sqrt{\left(G^{(1)}-G^{(2)}\right) / G^{(2)}}$ is the eigenvector of the non-degenerate wave of a generally non-unit length. Vectors $\mathbf{s}$ and $\mathbf{g}$ may be real- or complex-valued. Eq. (5) is a system of six quadratic equations in six unknowns: $\mathbf{s}=\left(s_{1}, s_{2}, s_{3}\right)^{T}$ and $\mathbf{g}=\left(g_{1}, g_{2}, g_{3}\right)^{T}$. The number of solutions is $2^{6}=64$. If we take into account that solutions of different signs: $\pm \mathbf{s}, \pm \mathbf{g}$, correspond to the same acoustic axis, the maximum number of acoustic axes is reduced from 64 to 16 .


Fig. 2. A sketch of slowness surfaces for $P$ and $S$ waves for isotropic media (a) and for $P, S 1$ and $S 2$ waves for weakly anisotropic media (b). The dots in (b) mark the positions of the acoustic axes.

Eliminating eigenvalues and eigenvectors in Eq. (4), we obtain [20]:

$$
\begin{align*}
& \left(\Gamma_{11}-\Gamma_{22}\right) \Gamma_{13} \Gamma_{23}-\Gamma_{12}\left(\Gamma_{13}^{2}-\Gamma_{23}^{2}\right)=0  \tag{6a}\\
& \left(\Gamma_{11}-\Gamma_{33}\right) \Gamma_{12} \Gamma_{23}-\Gamma_{13}\left(\Gamma_{12}^{2}-\Gamma_{23}^{2}\right)=0  \tag{6b}\\
& \left(\Gamma_{22}-\Gamma_{33}\right) \Gamma_{12} \Gamma_{13}-\Gamma_{23}\left(\Gamma_{12}^{2}-\Gamma_{13}^{2}\right)=0 . \tag{6c}
\end{align*}
$$

Eqs. (6a)-(6c) are suitable for calculating the acoustic axes numerically. They represent a system of sixth-order equations for three unknown components of the unit direction vector $\mathbf{n}: n_{1}, n_{2}$ and $n_{3}$. The three Eqs. (6a)-(6c) are not independent; hence we solve only two of them. We obtain 72 solutions, which are real- or complex-valued. Taking into account that $\pm \mathbf{n}$ describes the same direction, the number of directions reduces from 72 to 36 . Since the maximum number of acoustic axes is 16,20 solutions of Eqs. (6a)-(6c) are spurious and must be eliminated [27].

## 3. Calculation of acoustic axes in weak triclinic anisotropy

### 3.1. Weak anisotropy approximation

Weak triclinic anisotropy with elastic parameters $a_{i j k l}$ is obtained by perturbing an isotropic medium in the following way:

$$
\begin{equation*}
a_{i j k l}=a_{i j k l}^{0}+\varepsilon b_{i j k l}, \tag{7}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{i j k l}^{0}=\left(\alpha^{2}-2 \beta^{2}\right) \delta_{i j} \delta_{k l}+\beta^{2}\left(\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right)=\frac{\lambda}{\rho} \delta_{i j} \delta_{k l}+\frac{\mu}{\rho}\left(\delta_{i k} \delta_{j l}+\delta_{i l} \delta_{j k}\right) \tag{8}
\end{equation*}
$$

Parameters $\alpha$ and $\beta$ define the $P$ and $S$ velocities in an isotropic background medium, $\lambda$ and $\mu$ are the Lamé constants, $\rho$ is the medium density, tensor $b_{i j k l}$ defines the perturbations from isotropy into triclinic anisotropy, and $\varepsilon$ is a small quantity which measures anisotropy strength. In order to keep $\varepsilon$ within a reasonable range of values, tensor $b_{i j k l}$ should be scaled to have a norm comparable to the norm of $a_{i j k l}^{0}$.

For $\varepsilon=0$ in (7), weak anisotropy reduces to isotropy. The Christoffel tensor $\Gamma$ in (1) is degenerate with eigenvalues $G^{(1)}=\alpha^{2}$ and $G^{(2)}=G^{(3)}=\beta^{2}$, which are independent of direction $\mathbf{n}$. Consequently, the $P$ - and $S$-wave slowness surfaces are fully detached, $G^{(1)}>G^{(2)}$, and the $S 1$ and $S 2$ waves have coincident phase velocities in all directions $\mathbf{n}, G^{(2)}=G^{(3)}$. For a small non-zero $\varepsilon$, the $P$ wave remains detached from the $S 1$ and $S 2$ waves, but the global degeneracy of the $S 1$ and $S 2$ waves is removed, and the phase velocities of the $S 1$ and $S 2$ waves can coincide only in selected directions (see Fig. 2). Hence, under weak anisotropy we can observe just acoustic axes of the $S 1$ and $S 2$ waves. The acoustic axes of the $P$ and $S 1$ waves and the triple acoustic axes of the $P, S 1$ and $S 2$ waves cannot be observed.

### 3.2. Farra equations

Using perturbation theory for a degenerate Christoffel eigenvalue problem [30,32], we can express the difference between the $S$ waves in weak anisotropy as follows (Vavryčuk [8, Appendix A]):

$$
\begin{equation*}
\Delta G=G^{(2)}-G^{(3)}=\sqrt{\left[M^{(22)}-M^{(33)}\right]^{2}+4\left[M^{23}\right]^{2}} \tag{9}
\end{equation*}
$$

where

$$
\begin{align*}
M^{(K L)} & =m^{(K L)}+\frac{m^{(1 K)} m^{(1 L)}}{\beta^{2}-\alpha^{2}}, \quad K, L=2,3,  \tag{10}\\
m^{(r s)} & =\Gamma_{j k} g_{j}^{0(r)} g_{k}^{0(s)}, \quad r, s=1,2,3 . \tag{11}
\end{align*}
$$

Vector $\mathbf{g}^{0(1)}=\mathbf{n}$ is the $P$-wave polarization vector in the isotropic background, and vectors $\mathbf{g}^{0(2)}$ and $\mathbf{g}^{0(3)}$ form an arbitrary frame of perpendicular unit vectors lying in the plane with normal $\mathbf{n}$. The condition for an acoustic axis, $\Delta G=0$, yields in the first-order approximation:

$$
\begin{equation*}
m^{(22)}=m^{(33)} \quad \text { and } \quad m^{(23)}=0 \tag{12}
\end{equation*}
$$

Consequently, we obtain

$$
\begin{align*}
& a_{i j k l} n_{i} n_{l}\left(g_{j}^{0(2)} g_{k}^{0(2)}-g_{j}^{0(3)} g_{k}^{0(3)}\right)=0  \tag{13a}\\
& a_{i j k l} n_{i} n_{l} g_{j}^{0(2)} g_{k}^{0(3)}=0 \tag{13b}
\end{align*}
$$

Since Eqs. (13a)-(13b) are valid identically for the isotropic background, the equations hold also when elastic parameters $a_{i j k l}$ are substituted by perturbation parameters $b_{i j k l}$.

### 3.3. Linearization of the Darinskii equations

An alternative approach how to specify acoustic axes in weak triclinic anisotropy is to linearize Eq. (4). To do so we have to linearize equations for eigenvalues $G$ and eigenvectors $\mathbf{g}$ for the Christoffel tensor $\boldsymbol{\Gamma}$ along an acoustic axis [8,30]:

$$
\begin{align*}
G^{(1)} & =a_{i j k l} n_{i} n_{j} n_{k} n_{l}=\alpha^{2}+\varepsilon b_{i j k l} n_{i} n_{j} n_{k} n_{l}  \tag{14}\\
G^{(2)} & =G^{(3)}=\frac{1}{2}\left(\Gamma_{i i}-G^{(1)}\right)=\frac{1}{2}\left(a_{i j k i} n_{j} n_{k}-a_{l j k m} n_{j} n_{k} n_{l} n_{m}\right) \\
& =\beta^{2}+\frac{1}{2} \varepsilon b_{l j k m} n_{j} n_{k}\left(\delta_{l m}-n_{l} n_{m}\right)  \tag{15}\\
g_{i}^{(1)} & =n_{i}+\frac{1}{\alpha^{2}-\beta^{2}} \varepsilon b_{j l m k} n_{l} n_{m} n_{j}\left(\delta_{i k}-n_{i} n_{k}\right) \tag{16}
\end{align*}
$$

where we have used

$$
\begin{equation*}
\Gamma_{i i}=\operatorname{Tr}(\boldsymbol{\Gamma})=G^{(1)}+G^{(2)}+G^{(3)}=G^{(1)}+2 G^{(2)} \tag{17}
\end{equation*}
$$

Quantities $G^{(1)}$ and $G^{(2)}=G^{(3)}$ are eigenvalues of the Christoffel matrix $\Gamma$ for the $P$ waves and for the degenerate $S$ waves along the acoustic axis, and vector $\mathbf{g}^{(1)}$ is the $P$-wave polarization vector along the acoustic axis. Consequently, we obtain

$$
\begin{align*}
& G^{(1)}-G^{(2)}=\alpha^{2}-\beta^{2}+\frac{1}{2} \varepsilon b_{l j k m} n_{j} n_{k}\left(3 n_{l} n_{m}-\delta_{l m}\right)  \tag{18}\\
& g_{i}^{(1)} g_{l}^{(1)}=n_{i} n_{l}+\frac{1}{\alpha^{2}-\beta^{2}} \varepsilon b_{j o p k} n_{o} n_{p} n_{j}\left(\delta_{i k} n_{l}+\delta_{l k} n_{i}-2 n_{i} n_{k} n_{l}\right) \tag{19}
\end{align*}
$$

Inserting (18) and (19) into (4) and (5) we obtain

$$
\begin{align*}
& b_{p j k r} n_{p} n_{r}\left(2 \delta_{i j} \delta_{l k}-\delta_{j k} \delta_{i l}+n_{i} n_{l} \delta_{j k}+n_{j} n_{k} \delta_{i l}-2 n_{j} n_{l} \delta_{i k}\right. \\
& \left.\quad-2 n_{i} n_{j} \delta_{l k}+n_{j} n_{k} n_{i} n_{l}\right)=0, \quad i, l=1,2,3 \tag{20}
\end{align*}
$$

where we have omitted all higher-order terms of $\varepsilon$. Eq. (20) represents a system of 6 equations in three unknowns $n_{1}, n_{2}$ and $n_{3}$. Since $n_{1}, n_{2}$ and $n_{3}$ are the components of unit vector $\mathbf{n}$, only two components of $\mathbf{n}$ are independent.

### 3.4. Linearization of the Khatkevich equations

Acoustic axes in weak triclinic anisotropy can be calculated also linearizing Eqs. (6a)-(6c). Inserting (7) and (8) into Eqs. (6a)-(6c) we obtain [37]:

$$
\begin{align*}
& \varepsilon\left(\alpha^{2}-\beta^{2}\right)^{2} P_{1}\left(b_{i j k l}, n_{1}, n_{2}, n_{3}\right)+\varepsilon^{2}\left(\alpha^{2}-\beta^{2}\right) P_{2}\left(b_{i j k l}, n_{1}, n_{2}, n_{3}\right)+\varepsilon^{3} P_{3}\left(b_{i j k l}, n_{1}, n_{2}, n_{3}\right)=0  \tag{21a}\\
& \varepsilon\left(\alpha^{2}-\beta^{2}\right)^{2} Q_{1}\left(b_{i j k l}, n_{1}, n_{2}, n_{3}\right)+\varepsilon^{2}\left(\alpha^{2}-\beta^{2}\right) Q_{2}\left(b_{i j k l}, n_{1}, n_{2}, n_{3}\right)+\varepsilon^{3} Q_{3}\left(b_{i j k l}, n_{1}, n_{2}, n_{3}\right)=0  \tag{21b}\\
& \varepsilon\left(\alpha^{2}-\beta^{2}\right)^{2} R_{1}\left(b_{i j k l}, n_{1}, n_{2}, n_{3}\right)+\varepsilon^{2}\left(\alpha^{2}-\beta^{2}\right) R_{2}\left(b_{i j k l}, n_{1}, n_{2}, n_{3}\right)+\varepsilon^{3} R_{3}\left(b_{i j k l}, n_{1}, n_{2}, n_{3}\right)=0 \tag{21c}
\end{align*}
$$

where polynomials $P_{1}, Q_{1}$ and $R_{1}$ are of the first order, polynomials $P_{2}, Q_{2}$ and $R_{2}$ are of the second order, and $P_{3}, Q_{3}$ and $R_{3}$ are polynomials of the third order in $b_{i j k l}$.

If we assume $\varepsilon /\left(\alpha^{2}-\beta^{2}\right) \rightarrow 0$ (but not equal to zero), we can neglect the higher-order terms in $\varepsilon$ in Eqs. (21a)-(21c) and keep only the linear terms in $\varepsilon$. Then we obtain:

$$
\begin{align*}
& \sum_{i, j} p_{i j} u^{i} v^{j}=0,  \tag{22a}\\
& \sum_{i, j} q_{i j} u^{i} v^{j}=0,  \tag{22b}\\
& \sum_{i, j} r_{i j} u^{i} v^{j}=0, \tag{22c}
\end{align*}
$$

where $p_{i j}, q_{i j}$ and $r_{i j}$ are linear combinations of perturbation parameters $b_{i j k l}$, see Vavryčuk [37, Appendix B], and $u=n_{1} / n_{3}$ and $v=n_{2} / n_{3}$. Indices $i$ and $j$ run from zero to five and their sum is less than or equal to five. Eqs. (22) were derived under the assumption that no acoustic axis lies in the coordinate planes.

### 3.5. Calculation of acoustic axes

The most suitable equations for calculating acoustic axes in weak triclinic anisotropy are the Khatkevich linearized equations (22a)-(22c). The equations represent a system of three polynomial equations of the fifth order in two unknowns. However, the three Eqs. (22a)-(22c) are not independent; hence we solve only two of them. We obtain 16 real- or complexvalued solutions that lie out of coordinate planes. The real-valued solutions correspond to acoustic axes of homogeneous waves (with a real slowness vector). The complex-valued solutions correspond to acoustic axes of inhomogeneous waves (with a complex slowness vector); see Vavryčuk [27]. If the number of the solutions is less than 16, some of the acoustic axes lie in the coordinate planes. In this case, the coordinates should be rotated to move them out of the coordinate planes and the equations should be solved again.

The Darinskii linearized equations (20) represent a system of six equations in two unknowns. Similarly as for the Khatkevich equations, the six equations are not independent; hence we solve only two of them. However, the equations are of the sixth order in $\mathbf{n}$, and thus less suitable for calculating the acoustic axes than the Khatkevich equations. Similarly, the Farra equations (13a)-(13b) are not very suitable for calculating the acoustic axes. The equations contain vectors $\mathbf{g}^{\mathbf{0 ( 2 )}}$ and $\mathbf{g}^{0(3)}$, which should be eliminated. Using, for example, the following frame of vectors,

$$
\begin{equation*}
\mathbf{g}^{0(2)}=\frac{1}{\sqrt{n_{1}^{2}+n_{2}^{2}}}\left(n_{2},-n_{1}, 0\right)^{T} \quad \text { and } \quad \mathbf{g}^{0(3)}=\frac{1}{\sqrt{n_{1}^{2}+n_{2}^{2}}}\left(n_{1} n_{3}, n_{2} n_{3},-n_{1}^{2}-n_{2}^{2}\right)^{T} \tag{23}
\end{equation*}
$$

we obtain one polynomial equation of the sixth order in $\mathbf{n}$ from (13a), and one polynomial equation of the fifth order in $\mathbf{n}$ from (13b). Note that the exact directions of the acoustic axes are calculated by Eqs. (6a)-(6c), which represent a system of polynomial equations of the sixth order in $\mathbf{n}$. Hence, approximate equations by Darinskii and by Farra bring no advantage, because solving the approximate equations is as involved as solving the exact equations.

## 4. Inversion for anisotropy

The equations for the directions of acoustics axes in strongly anisotropic media are non-linear functions of elastic parameters $a_{i j k l}$. However, under weak anisotropy, the equations become linear. This is advantageous when solving the inverse problem: the determination of anisotropy from the directions of acoustic axes. Hereinafter, I shall formulate the inverse problem using the Farra equations, and linearized Darinskii and Khatkevich equations. On numerical examples, I shall calculate anisotropy from the directions of acoustic axes using all mentioned approaches and compare their accuracy.

### 4.1. Independent anisotropy parameters in the inversion

Eqs. (13), (20) and (22) can be used in the forward problem to seek the directions of acoustic axes from known anisotropy parameters, but they can also be used in the inverse problem to calculate anisotropy parameters from the known directions of acoustic axes. As expected, we cannot invert for the isotropic background, only perturbations into anisotropy can be calculated. In the inversion for anisotropy, the equations represent a system of linear homogeneous equations for coefficients of perturbation matrix $b_{i j k l}$. The matrix contains 21 parameters, but not all of them stand in the equations independently. Some of the parameters appear in combinations; hence the inversion can yield only the following 14 parameters:

$$
\begin{array}{llll}
\varepsilon^{(1)}=b_{45}, & \varepsilon^{(2)}=b_{46}, & \varepsilon^{(3)}=b_{56}, & \varepsilon^{(4)}=b_{44}-b_{66}, \\
\varepsilon^{(5)}=b_{55}-b_{66}, & \varepsilon^{(6)}=b_{34}-b_{14}, & \varepsilon^{(7)}=b_{34}-b_{24}, & \varepsilon^{(8)}=b_{35}-b_{15},
\end{array}
$$

$$
\begin{aligned}
& \varepsilon^{(12)}=\frac{1}{2}\left(b_{22}+b_{33}\right)-b_{23}-2 b_{66}, \quad \varepsilon^{(13)}=\frac{1}{2}\left(b_{11}+b_{33}\right)-b_{13}-2 b_{66}, \\
& \varepsilon^{(14)}=\frac{1}{2}\left(b_{11}+b_{22}\right)-b_{12}-2 b_{66} .
\end{aligned}
$$

Since the system of equations is homogeneous, one parameter must always be fixed to get a non-trivial solution. Thus we cannot retrieve absolute values, but only normalized values of the parameters. In other words, we cannot invert for strength of anisotropy.

### 4.2. Formulation of the inverse problem

Since only two equations are independent for each acoustic axis, we have to know at least seven directions of acoustic axes to obtain a properly determined system of equations. For seven acoustic axes, we obtain 14 equations. One equation can be omitted to get 13 equations for 13 unknowns. If we know a smaller number of axes, the system of equations is underdetermined and additional knowledge on anisotropy must be supplied for the system to be solvable. For example, we can add phase velocities or polarization vectors of $P$ waves at acoustic axes. If we know the directions of seven or more acoustic axes, the problem is overdetermined and anisotropy can be calculated using the standard least-squares method.

Eqs. (13), (20) or (22) can be expressed as follows:

$$
\begin{equation*}
\mathbf{A x}=\mathbf{c} \tag{25}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathbf{x}=\frac{1}{\varepsilon^{(14)}}\left[\varepsilon^{(1)}, \varepsilon^{(2)}, \varepsilon^{(3)}, \varepsilon^{(4)}, \varepsilon^{(5)}, \varepsilon^{(6)}, \varepsilon^{(7)}, \varepsilon^{(8)}, \varepsilon^{(9)}, \varepsilon^{(10)}, \varepsilon^{(11)}, \varepsilon^{(12)}, \varepsilon^{(13)}\right]^{T} \tag{26}
\end{equation*}
$$

is the $13 \times 1$ vector of unknown elastic parameters defining weak triclinic anisotropy, and $\mathbf{A}$ is the $2 N \times 13$ matrix and $\mathbf{c}$ is the $2 N \times 1$ vector, which are defined as follows

$$
\mathbf{A}=\left[\begin{array}{c}
\mathbf{A}^{(1)}  \tag{27}\\
\mathbf{A}^{(2)} \\
\vdots \\
\mathbf{A}^{(N)}
\end{array}\right], \quad \mathbf{c}=\left[\begin{array}{c}
\mathbf{c}^{(1)} \\
\mathbf{c}^{(2)} \\
\vdots \\
\mathbf{c}^{(N)}
\end{array}\right]
$$

$N$ denotes the number of acoustic axes. Quantities $\mathbf{A}^{(i)}$ and $\mathbf{c}^{(i)}$ are the $2 \times 13$ matrix and $2 \times 1$ vector, respectively, which are calculated for the $i$ th acoustic axis. Matrix $\mathbf{A}$ and vector $\mathbf{c}$ are specified in Appendix A for the Farra equations, in Appendix B for the linearized Darinskii equations, and in Appendix C for the linearized Khatkevich equations. The elastic parameters come out as

$$
\begin{equation*}
\mathbf{x}=\mathbf{A}^{-1} \mathbf{c} \tag{28}
\end{equation*}
$$

where $\mathbf{A}^{-1}$ stands for the least-squares inversion of matrix $\mathbf{A}$. As mentioned, Eq. (28) is solvable provided $N \geq 7$ ( $\mathbf{A}$ is the $14 \times 13$ matrix). If $N<7$, we have to complement Eq. (25) by other equations for the inversion for anisotropy to be feasible.

## 5. Numerical tests

In this section, I shall test the efficiency of the inversion for anisotropy using the above equations. I shall generate synthetic triclinic anisotropy and calculate the number and directions of acoustic axes. The directions of the axes will be inverted for anisotropy and the retrieved anisotropy parameters will be compared with the exact values.

The following tests will be performed: (1) Inversion for anisotropy of varying strength to determine the range of applicability of the inversion. (2) Inversion for anisotropy from approximate directions of the acoustic axes to test the sensitivity of the inversion to noise in the data. From this, the minimum accuracy required to measure the directions in experiments will be estimated. (3) Inversion for anisotropy performed using three different sets of equations to decide which method is the most efficient.

Weak triclinic anisotropy is calculated by Eqs. (7) and (8) with the $P$ and $S$ velocities of the isotropic background $\alpha=\sqrt{3} \mathrm{~km} / \mathrm{s}$, and $\beta=1 \mathrm{~km} / \mathrm{s}$, and with the following two perturbation matrices expressed in the Voigt notation (in $\mathrm{km}^{2} / \mathrm{s}^{2}$ ):

$$
\mathbf{b}^{(A)}=\left[\begin{array}{rrrrrr}
1.26 & -1.02 & -1.11 & 0.69 & 0.48 & 0.78  \tag{29}\\
& -1.14 & 0.81 & 0.51 & 1.17 & -0.66 \\
& & 0.00 & -0.24 & -0.84 & -1.38 \\
& & & 0.96 & -1.23 & 1.08 \\
& & & & 1.23 & -0.45 \\
& & & & & 0.00
\end{array}\right]
$$



Fig. 3. A polar plot of positions of acoustic axes for Anisotropy A (a) and Anisotropy B (b). The vertical axis is marked by the plus sign. For anisotropy parameters $b_{i j k l}$, see (29) and (30). Other parameters: $\alpha=\sqrt{3} \mathrm{~km} / \mathrm{s}, \beta=1 \mathrm{~km} / \mathrm{s}$, and $\varepsilon=0.1$.

$$
\mathbf{b}^{(B)}=\left[\begin{array}{rrrrrr}
-0.09 & 0.33 & -1.32 & -0.57 & 1.14 & -1.44  \tag{30}\\
& 0.21 & -0.24 & 0.81 & 1.44 & 1.50 \\
& & 0.00 & 0.87 & -0.18 & 0.00 \\
& & & 0.90 & -0.84 & 0.45 \\
& & & & -1.32 & -0.51 \\
& & & & & 0.00
\end{array}\right]
$$

The anisotropy model defined by Eq. (29) will be referred to as Anisotropy A, the anisotropy model defined by Eq. (30) will be referred to as Anisotropy B. Inserting (29) and (30) into (24) and (26) we obtain the following exact values of the anisotropy parameters, for which we shall invert

$$
\begin{align*}
\mathbf{x}^{\text {Exact }(A)}= & {[-1.139,1.000,-0.417,0.889,1.139,-0.861,-0.694,} \\
& -1.222,-1.861,-2.000,-0.667,-1.278,1.611]^{T},  \tag{31}\\
\mathbf{x}^{\text {Exact }(B)}= & {[-3.111,-1.667,1.889,-3.333,4.889,-5.333,-0.222,} \\
& 4.889,6.000,-5.333,5.556,-1.278,-4.722]^{T} . \tag{32}
\end{align*}
$$

Parameter $\varepsilon$ in (7) runs from 0.0001 to 0.1 , and it thus covers a wide range of anisotropy strength. Both anisotropy models have eight real-valued acoustic axes and their directions depend only very slightly on $\varepsilon$. For directions of acoustic axes for $\varepsilon=0.1$, see Fig. 3. We invert for anisotropy from the directions of all eight acoustic axes; hence, we use 16 independent equations in (25) and (28). Matrix A and vector care specified in Appendix A for the Farra equations, in Appendix B for the Darinskii equations, and in Appendix C for the Khatkevich equations. We invert from the exact directions of the acoustic axes as well as from the directions, which slightly deviate from the exact ones, to test the stability of the inversion. The deviations were generated randomly and varied within a wide range of values to mimic observations of different accuracy. For each specific accuracy, 2000 realizations of noisy directions of acoustic axes were generated and inverted for anisotropy. The relative error of each inversion was calculated as the average error $e$ of all components of vector $\mathbf{x}^{\text {Approx }}$,

$$
\begin{equation*}
e=\frac{1}{13} \sum_{i=1}^{13}\left|\frac{x_{i}^{\text {Exact }}-x_{i}^{\text {Approx }}}{x_{i}^{\text {Exact }}}\right| \cdot 100 \% . \tag{33}
\end{equation*}
$$

The relative error of the inversions for each accuracy of the directions of acoustic axes was calculated as the average error over all 2000 random realizations of noisy directions. The relative error of the inversion is studied as a function of the $P$-wave anisotropy strength $a^{P}$ calculated as

$$
\begin{equation*}
a^{P}=\frac{c^{\max }-c^{\min }}{c^{\max }+c^{\min }} \cdot 200 \% \tag{34}
\end{equation*}
$$

where $c^{\max }$ and $c^{\min }$ mean the maximum and minimum phase velocities of the $P$ wave over all directions of propagation.
Fig. 4 shows the relative error of anisotropy parameters as a function of anisotropy strength. The inversion is performed from exact directions of the acoustic axes, and we invert for anisotropy models A and B. As expected, if strength of anisotropy increases, the elastic parameters are retrieved with lower accuracy. Fig. 4 also shows that different equations used in the inversion yield a different accuracy. For both anisotropy models, the highest accuracy is achieved if we use the Farra equations. The Darinskii and Khatkevich equations yield less accurate results. However, the efficiency of the equations depends on the model and the results of the inversion cannot be easily generalized on the basis of just two anisotropy models. Moreover, the efficiency of the equations also depends on noise in the data. This is exemplified in Fig. 5, where the results of the inversion from inaccurate directions of acoustic axes are shown. The mean error in the directions is $0.1^{\circ}$. The figure


Fig. 4. Errors of calculated elastic parameters as a function of the $P$-wave anisotropy strength for the Farra (dotted line), Khatkevich (solid line) and Darinskii (dashed line) equations for Anisotropy A (a) and Anisotropy B (b). The inversion was performed using exact directions of acoustic axes.


Fig. 5. Errors of calculated elastic parameters as a function of the $P$-wave anisotropy strength for the Farra (dotted line), Khatkevich (solid line) and Darinskii (dashed line) equations for Anisotropy A (a) and Anisotropy B (b). The inversion was performed using approximate directions of acoustic axes. The average deviation $\Delta$ of the approximate directions was $0.1^{\circ}$.
indicates that noise in the directions of acoustic axes can prevent from accurate retrieving of anisotropy parameters. This applies to the whole range of anisotropy strength including extremely weak anisotropy. Fig. 5 also indicates that the accuracy of the inversions is remarkably different for Anisotropy A but it is very similar for all three approaches for Anisotropy B. In Anisotropy B, the Farra, Darinskii and Khatkevich equations yield accuracy which is roughly the same for the interval of anisotropy strength $0.01 \%-1 \%$. For higher anisotropy strength, the efficiency of the equations starts to differ.

Figs. 6-8 show a detailed comparison of the relative errors of anisotropy parameters as a function of anisotropy strength for models A and B. The errors are shown for the inversion from exact directions (curve a) and from directions of limited accuracy (curves b-f). Again, the errors of the inversion increase with increasing strength of anisotropy. If the anisotropy strength is higher than $10 \%$, the inversion fails. This applies to all three different sets of equations used in the inversion. This limitation follows from applying the weak anisotropy approximation. As expected, the errors also increase with decreasing accuracy of the directions of acoustic axes. If an average deviation between the exact and noisy directions of acoustic axes used in the inversion is higher than $1^{\circ}$ (curve f), the inversion yields errors ranging from $20 \%$ to $80 \%$, irrespective of the strength of anisotropy. This indicates rather high demands of the inversion on the accuracy of acoustic axes. To get results with accuracy better than $5 \%$, the acoustic axes must be determined with an accuracy of $0.1^{\circ}$ or better, and the anisotropy strength must be less than $1 \%$. To achieve such a high accuracy is not probably feasible in real experiments.

## 6. Conclusions

In principle, it is possible to invert for weak anisotropy parameters from directions of acoustic axes. The minimum number of acoustic axes, which can be inverted for weak anisotropy, is seven, and we can retrieve 13 combinations of


Fig. 6. Errors of calculated elastic parameters as a function of $P$-wave anisotropy strength for Anisotropy A (a) and Anisotropy B (b) using Farra equations. The inversion was performed using exact (curve a) and approximate (curves b-f) directions of acoustic axes. The average deviation of the approximate directions from the exact directions was (in degrees): 0.0043 (curve b), 0.017 (curve c), 0.069 (curve d), 0.27 (curve e), and 1.1 (curve f). The dashed line shows the error threshold of $10 \%$.


Fig. 7. Errors of calculated elastic parameters as a function of $P$-wave anisotropy strength for Anisotropy A (a) and Anisotropy B (b) using linearized Darinskii equations. The inversion was performed using exact (curve a) and approximate (curves b-f) directions of acoustic axes. The average deviation of the approximate directions from the exact directions was (in degrees): 0.0043 (curve b), 0.017 (curve c), 0.069 (curve d), 0.27 (curve e), and 1.1 (curve f). The dashed line shows the error threshold of $10 \%$.


Fig. 8. Errors of calculated elastic parameters as a function of $P$-wave anisotropy strength for Anisotropy A (a) and Anisotropy B (b) using linearized Khatkevich equations. The inversion was performed using exact (curve a) and approximate (curves b-f) directions of acoustic axes. The average deviation of the approximate directions from the exact directions was (in degrees): 0.0043 (curve b), 0.017 (curve c), 0.069 (curve d), 0.27 (curve e), and 1.1 (curve f). The dashed line shows the error threshold of $10 \%$.
elastic parameters. Under the weak anisotropy approximation, the directions of acoustic axes are insensitive to strength of anisotropy; hence we cannot invert for absolute values of weak anisotropy parameters, but only for their relative values. Numerical tests have shown that the inversion is applicable to very weak anisotropy with strength of less than $5 \%$, provided that the acoustic axes used in the inversion are determined with an accuracy of $0.1^{\circ}$ or better. In this case, the inversion yields an average error of elastic parameters less than $10 \%$. The tests have also revealed significant differences between different linearized equations: the Farra equations, the linearized Darinskii equations or the linearized Khatkevich equations. The efficiency of the equations depends on the positions of the acoustic axes on the sphere and on their accuracy. This dependence is complicated, and no simple rule indicates which method is preferable for the inversion.

If the studied weak anisotropy possesses less than seven real-valued acoustic axes or if they are not measured with a sufficient accuracy, we cannot invert for the total set of 13 combinations of anisotropy parameters from acoustic axes only. Since both conditions are quite restrictive being not fulfilled in many cases, we have to incorporate additional equations in the inversion. For example, we can incorporate measurements of phase velocities in selected directions. The combination of the directions of the acoustic axes with measurements of other attributes of elastic waves in the inversion is advantageous also for the sake of improving the accuracy of the weak anisotropy parameters and retrieving other anisotropy parameters which cannot, in principle, be obtained from acoustic axes including the strength of anisotropy.

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## Appendix A. The Farra equations

Using Eqs. (13) we obtain the following components of submatrix $\mathbf{A}^{(i)}$ and of vector $\mathbf{c}^{(i)}$ specified for the $i$ th acoustic axis:

$$
\begin{aligned}
& A_{1}^{1}=n_{3}\left(n_{1}^{4}-n_{1}^{2} n_{3}^{2}-n_{2}^{4}+n_{2}^{2} n_{3}^{2}\right), \\
& A_{3}^{1}=n_{1}\left(n_{1}^{4}-n_{1}^{2} n_{3}^{2}-n_{2}^{4}+3 n_{2}^{2} n_{3}^{2}\right), \\
& A_{5}^{1}=n_{1} n_{2} n_{3}\left(-n_{1}^{2}-n_{2}^{2}+n_{3}^{2}\right), \\
& A_{7}^{1}=n_{1} n_{2}^{2}\left(-n_{1}^{2}-n_{2}^{2}+2 n_{3}^{2}\right), \\
& A_{9}^{1}=n_{2}\left(-n_{1}^{4}-n_{1}^{2} n_{2}^{2}+n_{1}^{2} n_{3}^{2}-n_{2}^{2} n_{3}^{2}\right), \\
& A_{11}^{1}=n_{2}^{2} n_{3}\left(3 n_{1}^{2}-n_{2}^{2}\right), \\
& A_{13}^{1}=n_{1} n_{2} n_{3}\left(n_{1}^{2}+n_{2}^{2}\right), \\
& A_{1}^{2}=2 n_{1} n_{2}\left(-n_{1}^{4}-2 n_{1}^{2} n_{2}^{2}+2 n_{1}^{2} n_{3}^{2}-n_{2}^{4}+2 n_{2}^{2} n_{3}^{2}-n_{3}^{4}-n_{3}^{2}\right), \\
& A_{2}^{2}=2 n_{1} n_{3}\left(2 n_{1}^{2} n_{2}^{2}+n_{1}^{2}+2 n_{2}^{4}-2 n_{2}^{2} n_{3}^{2}-n_{2}^{2}\right), \\
& A_{3}^{2}=2 n_{2} n_{3}\left(2 n_{1}^{4}+2 n_{1}^{2} n_{2}^{2}-2 n_{1}^{2} n_{3}^{2}-n_{1}^{2}+n_{2}^{2}\right), \\
& A_{4}^{2}=-n_{1}^{4} n_{2}^{2}-2 n_{1}^{2} n_{2}^{4}+2 n_{1}^{2} n_{2}^{2} n_{3}^{2}+n_{1}^{2} n_{3}^{2}-n_{2}^{6}+2 n_{2}^{4} n_{3}^{2}-n_{2}^{2} n_{3}^{4}, \\
& A_{5}^{2}=-n_{1}^{6}-2 n_{1}^{4} n_{2}^{2}+2 n_{1}^{4} n_{3}^{2}-n_{1}^{2} n_{2}^{4}+2 n_{1}^{2} n_{2}^{2} n_{3}^{2}-n_{1}^{2} n_{3}^{4}+n_{2}^{2} n_{3}^{2}, \\
& A_{6}^{2}=2 n_{1}^{2} n_{2} n_{3}\left(-n_{1}^{2}-n_{2}^{2}+n_{3}^{2}+1\right), \\
& A_{8}^{2}=2 n_{1} n_{3}\left(-n_{1}^{4}-n_{1}^{2} n_{2}^{2}+n_{1}^{2} n_{3}^{2}-n_{2}^{2}\right), \\
& A_{10}^{2}=2 n_{1} n_{2}\left(2 n_{1}^{2} n_{3}^{2}+n_{1}^{2}-n_{2}^{2}\right), \\
& A_{12}^{2}=-2 n_{2}^{2} n_{3}^{2}\left(n_{1}^{2}+n_{2}^{2}\right),
\end{aligned}
$$

$$
A_{2}^{1}=n_{2}\left(n_{1}^{4}-3 n_{1}^{2} n_{3}^{2}-n_{2}^{4}+n_{2}^{2} n_{3}^{2}\right)
$$

$$
A_{4}^{1}=n_{1} n_{2} n_{3}\left(n_{1}^{2}+n_{2}^{2}-n_{3}^{2}\right),
$$

$$
A_{6}^{1}=n_{1}\left(n_{1}^{2} n_{2}^{2}+n_{1}^{2} n_{3}^{2}+n_{2}^{4}-n_{2}^{2} n_{3}^{2}\right)
$$

$$
A_{8}^{1}=n_{1}^{2} n_{2}\left(n_{1}^{2}+n_{2}^{2}-2 n_{3}^{2}\right),
$$

$$
A_{10}^{1}=n_{1}^{2} n_{3}\left(n_{1}^{2}-3 n_{2}^{2}\right),
$$

$$
A_{12}^{1}=-n_{1} n_{2} n_{3}\left(n_{1}^{2}+n_{2}^{2}\right),
$$

$$
A_{6}^{2}=2 n_{1}^{2} n_{2} n_{3}\left(-n_{1}^{2}-n_{2}^{2}+n_{3}^{2}+1\right), \quad A_{7}^{2}=2 n_{2} n_{3}\left(-n_{1}^{2} n_{2}^{2}-n_{1}^{2}-n_{2}^{4}+n_{2}^{2} n_{3}^{2}\right)
$$

$$
A_{9}^{2}=2 n_{1} n_{2}^{2} n_{3}\left(-n_{1}^{2}-n_{2}^{2}+n_{3}^{2}+1\right),
$$

$$
A_{11}^{2}=2 n_{1} n_{2}\left(-n_{1}^{2}+2 n_{2}^{2} n_{3}^{2}+n_{2}^{2}\right)
$$

$$
A_{13}^{2}=-2 n_{1}^{2} n_{3}^{2}\left(n_{1}^{2}+n_{2}^{2}\right)
$$

$$
\begin{aligned}
& \mathbf{A}^{(i)}=\left[\begin{array}{lllllllllllll}
A_{1}^{1} & A_{2}^{1} & A_{3}^{1} & A_{4}^{1} & A_{5}^{1} & A_{6}^{1} & A_{7}^{1} & A_{8}^{1} & A_{9}^{1} & A_{10}^{1} & A_{11}^{1} & A_{12}^{1} & A_{13}^{1} \\
A_{1}^{2} & A_{2}^{2} & A_{3}^{2} & A_{4}^{2} & A_{5}^{2} & A_{6}^{2} & A_{7}^{2} & A_{8}^{2} & A_{9}^{2} & A_{10}^{2} & A_{11}^{2} & A_{12}^{2} & A_{13}^{2}
\end{array}\right], \\
& \mathbf{c}^{(i)}=\left[\begin{array}{c}
-n_{1} n_{2} n_{3}\left(n_{1}^{2}-n_{2}^{2}\right) \\
-2 n_{1}^{2} n_{2}^{2}\left(n_{3}^{2}+1\right)
\end{array}\right],
\end{aligned}
$$

where superscript $i$ denotes the sequential number of the acoustic axis, and $\mathbf{n}$ is the direction vector of this axis. Inserting submatrices $\mathbf{A}^{(i)}$ and vectors $\mathbf{c}^{(i)}$ into (27) we construct matrix $\mathbf{A}$ and vector $\mathbf{c}$ required in the system of linear equations (28).

## Appendix B. The linearized Darinskii equations

Using Eq. (20), we obtain the following components of submatrix $\mathbf{A}^{(i)}$ and of vector $\mathbf{c}^{(i)}$ specified for the $i$ th acoustic axis:

$$
\begin{array}{ll}
A_{1}^{1}=2 n_{1} n_{2}\left(4 n_{1}^{2} n_{3}^{2}+n_{1}^{2}-1\right), & A_{2}^{1}=2 n_{1} n_{3}\left(4 n_{1}^{2} n_{2}^{2}+n_{1}^{2}-1\right), \\
A_{3}^{1}=2 n_{2} n_{3}\left(4 n_{1}^{4}-3 n_{1}^{2}+1\right), & A_{4}^{1}=4 n_{1}^{2} n_{2}^{2} n_{3}^{2}+n_{1}^{2} n_{2}^{2}+n_{1}^{2} n_{3}^{2}+4 n_{2}^{2} n_{3}^{2}-n_{2}^{2}-n_{3}^{2}, \\
A_{5}^{1}=4 n_{1}^{4} n_{3}^{2}+n_{1}^{4}-3 n_{1}^{2} n_{3}^{2}-n_{1}^{2}+n_{3}^{2}, & A_{6}^{1}=4 n_{1}^{2} n_{2} n_{3}\left(-n_{1}^{2}+1\right), \\
A_{7}^{1}=2 n_{2} n_{3}\left(-2 n_{1}^{2} n_{2}^{2}-n_{1}^{2}-2 n_{2}^{2}+1\right), & A_{8}^{1}=2 n_{1} n_{3}\left(-2 n_{1}^{4}+3 n_{1}^{2}-1\right), \\
A_{9}^{1}=-4 n_{1}^{3} n_{2}^{2} n_{3}, & A_{10}^{1}=2 n_{1} n_{2}\left(-2 n_{1}^{4}+3 n_{1}^{2}-1\right), \\
A_{11}^{1}=2 n_{1} n_{2}\left(-2 n_{1}^{2} n_{2}^{2}-n_{1}^{2}+1\right), & A_{12}^{1}=-2 n_{2}^{2} n_{3}^{2}\left(n_{1}^{2}+1\right), \\
A_{13}^{1}=2 n_{1}^{2} n_{3}^{2}\left(-n_{1}^{2}+1\right), & A_{1}^{2}=2 n_{1} n_{2}\left(4 n_{2}^{2} n_{3}^{2}+n_{2}^{2}-1\right), \\
A_{2}^{2}=2 n_{1} n_{3}\left(4 n_{2}^{4}-3 n_{2}^{2}+1\right), & A_{3}^{2}=2 n_{2} n_{3}\left(4 n_{1}^{2} n_{2}^{2}+n_{2}^{2}-1\right), \\
A_{4}^{2}=4 n_{2}^{4} n_{3}^{2}+n_{2}^{4}-3 n_{2}^{2} n_{3}^{2}-n_{2}^{2}+n_{3}^{2}, & A_{5}^{2}=4 n_{1}^{2} n_{2}^{2} n_{3}^{2}+n_{1}^{2} n_{2}^{2}+4 n_{1}^{2} n_{3}^{2}+n_{2}^{2} n_{3}^{2}-n_{1}^{2}-n_{3}^{2}, \\
A_{6}^{2}=-4 n_{1}^{2} n_{2}^{3} n_{3}, & A_{7}^{2}=2 n_{2} n_{3}\left(-2 n_{2}^{4}+3 n_{2}^{2}-1\right), \\
A_{8}^{2}=2 n_{1} n_{3}\left(-2 n_{1}^{2} n_{2}^{2}-2 n_{1}^{2}-n_{2}^{2}+1\right), & A_{9}^{2}=4 n_{1} n_{2}^{2} n_{3}\left(-n_{2}^{2}+1\right), \\
A_{10}^{2}=2 n_{1} n_{2}\left(-2 n_{1}^{2} n_{2}^{2}-n_{2}^{2}+1\right), & A_{11}^{2}=2 n_{1} n_{2}\left(-2 n_{2}^{4}+3 n_{2}^{2}-1\right), \\
A_{12}^{2}=2 n_{2}^{2} n_{3}^{2}\left(-n_{2}^{2}+1\right), & A_{13}^{2}=-2 n_{1}^{2} n_{3}^{2}\left(n_{2}^{2}+1\right),
\end{array}
$$

$$
\mathbf{A}^{(i)}=\left[\begin{array}{lllllllllllll}
A_{1}^{1} & A_{2}^{1} & A_{3}^{1} & A_{4}^{1} & A_{5}^{1} & A_{6}^{1} & A_{7}^{1} & A_{8}^{1} & A_{9}^{1} & A_{10}^{1} & A_{11}^{1} & A_{12}^{1} & A_{13}^{1} \\
A_{1}^{2} & A_{2}^{2} & A_{3}^{2} & A_{4}^{2} & A_{5}^{2} & A_{6}^{2} & A_{7}^{2} & A_{8}^{2} & A_{9}^{2} & A_{10}^{2} & A_{11}^{2} & A_{12}^{2} & A_{13}^{2}
\end{array}\right],
$$

$$
\mathbf{c}^{(i)}=\left[\begin{array}{l}
2 n_{1}^{2} n_{2}^{2}\left(n_{1}^{2}-1\right) \\
2 n_{1}^{2} n_{2}^{2}\left(n_{2}^{2}-1\right)
\end{array}\right],
$$

where superscript $i$ denotes the sequential number of the acoustic axis, and $\mathbf{n}$ is the direction vector of this axis. Inserting submatrices $\mathbf{A}^{(i)}$ and vectors $\mathbf{c}^{(i)}$ into (27) we construct matrix $\mathbf{A}$ and vector $\mathbf{c}$ required in the system of linear equations (28).

## Appendix C. The linearized Khatkevich equations

Using Eqs. (22) and specifying coefficients $p_{i j}, q_{i j}$ and $r_{i j}$ according to Appendix B of Vavryčuk [37], we obtain the following components of submatrix $\mathbf{A}^{(i)}$ and of vector $\mathbf{c}^{(i)}$ specified for the $i$ th acoustic axis:

$$
A_{2}^{1}=n_{2}\left(n_{1}^{4}-3 n_{1}^{2} n_{3}^{2}-n_{2}^{4}+n_{2}^{2} n_{3}^{2}\right)
$$

$$
A_{4}^{1}=n_{1} n_{2} n_{3}\left(n_{1}^{2}+n_{2}^{2}-n_{3}^{2}\right),
$$

$$
A_{6}^{1}=n_{1}\left(n_{1}^{2} n_{2}^{2}+n_{1}^{2} n_{3}^{2}+n_{2}^{4}-n_{2}^{2} n_{3}^{2}\right),
$$

$$
A_{8}^{1}=n_{1}^{2} n_{2}\left(n_{1}^{2}+n_{2}^{2}-2 n_{3}^{2}\right),
$$

$$
A_{10}^{1}=n_{1}^{2} n_{3}\left(n_{1}^{2}-3 n_{2}^{2}\right)
$$

$$
A_{12}^{1}=-n_{1} n_{2} n_{3}\left(n_{1}^{2}+n_{2}^{2}\right),
$$

$$
A_{2}^{2}=n_{2}\left(n_{1}^{4}-n_{1}^{2} n_{2}^{2}+n_{2}^{2} n_{3}^{2}-n_{3}^{4}\right),
$$

$$
A_{4}^{2}=n_{1} n_{2} n_{3}\left(n_{1}^{2}-n_{2}^{2}+n_{3}^{2}\right),
$$

$$
A_{6}^{2}=n_{1}\left(n_{1}^{2} n_{2}^{2}+n_{1}^{2} n_{3}^{2}-n_{2}^{2} n_{3}^{2}+n_{3}^{4}\right),
$$

$$
A_{8}^{2}=n_{1}^{2} n_{2}\left(n_{1}^{2}-3 n_{3}^{2}\right),
$$

$$
A_{10}^{2}=n_{1}^{2} n_{3}\left(n_{1}^{2}-2 n_{2}^{2}+n_{3}^{2}\right),
$$

$$
\begin{aligned}
& A_{1}^{1}=n_{3}\left(n_{1}^{4}-n_{1}^{2} n_{3}^{2}-n_{2}^{4}+n_{2}^{2} n_{3}^{2}\right), \\
& A_{3}^{1}=n_{1}\left(n_{1}^{4}-n_{1}^{2} n_{3}^{2}-n_{2}^{4}+3 n_{2}^{2} n_{3}^{2}\right) \text {, } \\
& A_{5}^{1}=n_{1} n_{2} n_{3}\left(-n_{1}^{2}-n_{2}^{2}+n_{3}^{2}\right) \text {, } \\
& A_{7}^{1}=n_{1} n_{2}^{2}\left(-n_{1}^{2}-n_{2}^{2}+2 n_{3}^{2}\right) \text {, } \\
& A_{9}^{1}=n_{2}\left(-n_{1}^{4}-n_{1}^{2} n_{2}^{2}+n_{1}^{2} n_{3}^{2}-n_{2}^{2} n_{3}^{2}\right) \text {, } \\
& A_{11}^{1}=n_{2}^{2} n_{3}\left(3 n_{1}^{2}-n_{2}^{2}\right) \text {, } \\
& A_{13}^{1}=n_{1} n_{2} n_{3}\left(n_{1}^{2}+n_{2}^{2}\right) \text {, } \\
& A_{1}^{2}=n_{3}\left(n_{1}^{4}-3 n_{1}^{2} n_{2}^{2}+n_{2}^{2} n_{3}^{2}-n_{3}^{4}\right) \text {, } \\
& A_{3}^{2}=n_{1}\left(n_{1}^{4}-n_{1}^{2} n_{2}^{2}+3 n_{2}^{2} n_{3}^{2}-n_{3}^{4}\right) \text {, } \\
& A_{5}^{2}=2 n_{1} n_{2} n_{3}\left(-n_{1}^{2}+n_{3}^{2}\right) \text {, } \\
& A_{7}^{2}=-n_{1} n_{2}^{2}\left(n_{1}^{2}+n_{3}^{2}\right) \text {, } \\
& A_{9}^{2}=n_{2}\left(-n_{1}^{4}+n_{3}^{4}\right) \text {, } \\
& A_{11}^{2}=n_{2}^{2} n_{3}\left(n_{1}^{2}+n_{3}^{2}\right) \text {, } \\
& A_{13}^{2}=n_{1} n_{2} n_{3}\left(n_{1}^{2}-n_{3}^{2}\right) \text {, } \\
& \mathbf{A}^{(i)}=\left[\begin{array}{lllllllllllll}
A_{1}^{1} & A_{2}^{1} & A_{3}^{1} & A_{4}^{1} & A_{5}^{1} & A_{6}^{1} & A_{7}^{1} & A_{8}^{1} & A_{9}^{1} & A_{10}^{1} & A_{11}^{1} & A_{12}^{1} & A_{13}^{1} \\
A_{1}^{2} & A_{2}^{2} & A_{3}^{2} & A_{4}^{2} & A_{5}^{2} & A_{6}^{2} & A_{7}^{2} & A_{8}^{2} & A_{9}^{2} & A_{10}^{2} & A_{11}^{2} & A_{12}^{2} & A_{13}^{2}
\end{array}\right], \\
& \mathbf{c}^{(i)}=\left[\begin{array}{l}
-n_{1} n_{2} n_{3}\left(n_{1}^{2}-n_{2}^{2}\right) \\
-n_{1} n_{2} n_{3}\left(n_{1}^{2}+n_{3}^{2}\right)
\end{array}\right],
\end{aligned}
$$

where superscript $i$ denotes the sequential number of the acoustic axis, and $\mathbf{n}$ is the direction vector of this axis. Inserting submatrices $\mathbf{A}^{(i)}$ and vectors $\mathbf{c}^{(i)}$ into (27) we construct matrix $\mathbf{A}$ and vector $\mathbf{c}$ required in the system of linear equations (28).

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